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Perturbation Theory of Eigenvalue Problems

F. RELLICH

Research in the Field of Perturbation Theory
and Linear Operators

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PERTURBATION THEORY OF EIGENVALUE PROBLEMS

by

F. Rellich

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PERTURBATION THEORY OF EIGENVALUE PROBLEMS

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The present report represents notes of lectures on the theory of perturbations delivered by Professor F. Rellich at New York University, fall 1953. These notes were prepared with the assistance of Dr. J. Berkowitz.

The notes may serve as the first introductory text to the mathematical theory of perturbation of linear operators.

So far there does not exist any text within this field in any language. Various original research papers are scattered in the mathematical and physical journals; moreover a good many--and some of the most important--of these papers are not in English. Some works on perturbation theory are concerned with rather abstract operator-theoretic questions for their intrinsic interest, while others are concerned with application of the perturbation theory formalism to concrete problems arising in the physical sciences. The presentation here strikes a middle course in formulating the abstract theory in such a way that it lends itself to applications involving differential operators. A variety of special problems are considered to indicate the nature of typical results and to offer some unsolved mathematical problems that have not as yet yielded to the general theory.

Among the specific features of the work presented in these notes we mention the following:

A special example of non-analytic perturbation is studied in detail to demonstrate how perturbation considerations can sometimes be used to shed light on the physical significance of some aspect of a mathematical model of a physical system (cf. pp. 24-35).

Possible extension of perturbation theory to problems involving more than one perturbation parameter is studied, and some of the difficulties therein are described.

The new results obtained by F. Rellich and presented in these notes are the following:

It is proved (cf. p. 55) that the least eigenvalue is a convex function of a linear perturbation parameter. Garabedian and Schiffer had made use of this fact in certain special cases, but Rellich observed that the result obtains in general with virtually the same proof. Although in this report a proof is given only in the finite dimensional case, Rellich has proved the appropriate analogous statement for Hermitian operators in infinite dimensional Hilbert space; a paper on this result will be forthcoming.

A further new result (cf. pp. 56-69) is this: Suppose $A(\varepsilon)$ is a Hermitian operator on a finite dimensional Hilbert space, and if $A(\varepsilon)$ has a continuous derivative for real values of ε, then eigenvalues of $A(\varepsilon)$ can be arranged so that they are continuously differentiable functions of ε. The proof of this fact involves rather intricate real-variable considerations.

The inequality $c\int r^{-4}|u|^2\,d\tau \leq \int |\Delta u|^2\,d\tau$ was discovered and investigated by F. Rellich (cf. pp. 121-134). The best constant c for which the inequality holds is given by $c = \frac{(m-4)^2m^2}{16}$, where m is the dimension of the space of the independent variables, except in the case $m = 2$. The proof involves expanding u in terms of spherical harmonics of dimension m, and then a careful examination of the terms. The method of proof has quite general interest. A host of related inequalities can be established, which have bearing on the theory of partial differential equations. The motivation for the study of this inequality in the context of these notes is to verify a criterion for the "regularity" of a perturbation in a problem arising from the theory of elasticity.

K. O. Friedrichs

PERTURBATION THEORY OF EIGENVALUE PROBLEMS

Introduction

Perturbation methods attempt to solve a given problem by approximating it by simpler problems whose solutions are more or less explicitly known.

In eigenvalue problems the perturbation method yields numerical results comparatively quickly provided you are satisfied with approximations of low order. However, even in problems which appear to be very simple, it might be difficult to ascertain whether or not the method applied would converge if continued ad infinitum or to estimate the error incurred at stopping at a certain order of approximation. Sometimes the method obviously does not converge--at least not in the usual sense; then there is the problem of trying to interpret the results computed, if they have any significance at all. The following examples illustrate the mathematical problems involved in these questions.

§1. A small perturbation parameter does not mean a small perturbation.

We consider the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|}\, u(y)dy = \lambda u(x) , \qquad -\infty < x < +\infty \quad .$$

By a point eigenvalue we mean any number λ for which there exists a (possibly complex-valued) solution $u(x) \not\equiv 0$ of this integral equation such that $\int_{-\infty}^{\infty} |u|^2 dx < \infty$. (Integration here is understood to be in the sense of Lebesgue.) The function $u(x)$ is called an eigenfunction belonging to λ. We use the name "point eigenvalue" rather than just "eigenvalue" to emphasize the fact that we do not mean the points of the

continuous spectrum (which happens to be absent in this simple example). It is a simple matter to find all the point eigenvalues in this example. Suppose λ is an eigenvalue; then

$$\lambda u(x) = ce^{-|x|}$$

where

$$c = \int_{-\infty}^{\infty} e^{-|y|} u(y)dy \quad .$$

Hence

$$\lambda \int_{-\infty}^{\infty} e^{-|x|} u(x)dx = c \int_{-\infty}^{\infty} e^{-2|x|} dx = c \quad ,$$

and consequently

$$\lambda c = c \quad .$$

If $c \neq 0$ then $\lambda = 1$ and $\phi_o(x) = e^{-|x|}$ is an eigenfunction belonging to $\lambda = 1$. If $c = 0$ then $\lambda = 0$, for otherwise $u(x) \equiv 0$ which is not possible for an eigenfunction. Thus every function $\phi(x)$ with $\phi(x) \not\equiv 0$, $\int_{-\infty}^{\infty} |\phi|^2 dx < \infty$, and $\int_{-\infty}^{\infty} e^{-|x|} \phi(x)dx = 0$ is an eigenfunction belonging to the point eigenvalue $\lambda = 0$. An arbitrary function $f(x)$ with $\int_{-\infty}^{\infty} |f|^2 dx < \infty$ can be "expanded" in the form

$$f(x) = f_o\phi_o(x) + \phi(x) \quad ,$$

with

$$f_o = \int_{-\infty}^{\infty} \phi_o(x)f(x)dx \quad ,$$

where ϕ_o and ϕ are eigenfunctions belonging to $\lambda = 1$ and to $\lambda = 0$, respectively. The point eigenvalue $\lambda = 1$ is simple (non-degenerate) whereas $\lambda = 0$ is of infinite multiplicity.

We now turn to the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)dy + \varepsilon xu(x) = \lambda u(x) \quad , \quad -\infty < x < +\infty \quad ,$$

with ε a given real number. Assuming ε small, we seek a point eigenvalue $\lambda = \lambda(\varepsilon)$ and a corresponding eigenfunction $u = u(x;\varepsilon)$ of the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots \, , \qquad \lambda = 1 \quad ;$$

$$u = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots \, , \qquad \phi_0 = e^{-|x|} \quad .$$

If we introduce the abbreviation

$$Au = \int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)dy \quad , \quad Bu = xu(x) \quad ,$$

we have

$$(A + \varepsilon B)(\phi_0 + \varepsilon\phi_1 + \dots) = (\lambda_0 + \varepsilon\lambda_1 + \dots)(\phi_0 + \varepsilon\phi_1 + \dots) \quad .$$

If the convergence of $\phi_0 + \varepsilon\phi_1 + \dots$ is good enough to permit the term by term multiplication

$$(A + \varepsilon B)(\phi_0 + \varepsilon\phi_1 + \dots) = A\phi_0 + \varepsilon(B\phi_0 + A\phi_1) + \dots \quad ,$$

then we obtain the equations:

$$A\phi_0 - \lambda_0\phi_0 = 0 \quad ,$$

$$A\phi_1 - \lambda_0\phi_1 = \lambda_1\phi_0 - B\phi_0 \quad ,$$

$$A\phi_2 - \lambda_0\phi_2 = \lambda_2\phi_0 + \lambda_1\phi_1 - B\phi_1 \quad ,$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$A\phi_n - \lambda_0\phi_n = \lambda_n\phi_0 + \lambda_{n-1}\phi_1 + \dots + \lambda_1\phi_{n-1} - B\phi_{n-1}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The first equation provides no new information; it is satisfied by $\lambda_o = 1$, $\phi_o = e^{-|x|}$. The remaining equations are of the form

$$A\phi_n - \lambda_o\phi_n = f \quad , \qquad n = 1,2,\ldots \quad ;$$

consequently we have

$$(\phi_o, A\phi_n - \lambda\phi_n) = (\phi_o, f) \quad ,$$

using here the "inner product"

$$(g,f) = \int_{-\infty}^{\infty} \overline{g(x)} f(x) dx \quad .$$

Evidently

$$(\phi_o, A\phi_n) = (A\phi_o, \phi_n)$$

because the kernel $e^{-|x|-|y|}$ of the integral operator A is symmetric in x and y. Thus

$$(\phi_o, f) = (\phi, A\phi_n) - \lambda(\phi_o, \phi_n) = 0 \quad ,$$

and therefore

$$\lambda_n(\phi_o,\phi_o) + \lambda_{n-1}(\phi_o,\phi_1) + \ldots + \lambda_1(\phi_o,\phi_{n-1}) - (\phi_o, B\phi_{n-1}) = 0 \quad ,$$

$$n = 1,2,\ldots \quad .$$

For n = 1, we find

$$\lambda_1 = (\phi_o, B\phi_o) \quad ,$$

or in words:

The first order approximation to the eigenvalue is the mean value of the perturbation operator B with respect to the unperturbed normalized eigenfunction ϕ_o.

In our special case we have

$$\lambda_1 = \int_{-\infty}^{\infty} x e^{-2|x|} dx = 0 \quad .$$

The first order approximation $\phi_1(x)$ can be determined from the equation

$$A\phi_1 - \lambda_0\phi_1 = \lambda_1\phi_0 - B\phi_0 \quad ;$$

the right hand side is now known. We find

$$c_1 e^{-|x|} - \phi_1 = -xe^{-|x|}$$

where

$$c_1 = \int_{-\infty}^{\infty} e^{-|x|} \phi_1(x)dx \quad ,$$

and thus

$$\phi_1 = (x + c_1)e^{-|x|} \quad .$$

Evidently this function is a solution of $A\phi_1 - \lambda_0\phi_1 = \lambda_1\phi_0 - B\phi_0$ no matter what the value of c_1. If we require that the function $u = \phi_0 + \varepsilon\phi_1 + \ldots$ be normalized, we must have

$$1 = (\phi_0 + \varepsilon\phi_1 + \ldots, \phi_0 + \varepsilon\phi_1 + \ldots) \quad ,$$

which gives rise to the following infinitely many equations:

$$1 = (\phi_0, \phi_0) \quad ,$$

$$0 = (\phi_0, \phi_1) + (\phi_1, \phi_0) \quad ,$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$0 = (\phi_0, \phi_n) + (\phi_1, \phi_{n-1}) + \ldots + (\phi_n, \phi_0) \quad ,$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

In addition we require all the functions $\phi_1, \phi_2, \ldots$ to be real. Then

$$0 = (\phi_0, \phi_1) + (\phi_1, \phi_0) = 2(\phi_0, \phi_1) \quad ,$$

and hence

$$0 = (\phi_0, \phi_1) = \int_{-\infty}^{\infty} (x + c_1)e^{-2|x|} dx = c_1 \quad .$$

Thus

$$\phi_1(x) = xe^{-|x|} \quad .$$

Since

$$\lambda_2(\phi_o,\phi_o) + \lambda_1(\phi_o,\phi_1) - (\phi_o,B\phi_1) = 0 \quad ,$$

we obtain

$$\lambda_2 = \frac{1}{2} \quad .$$

Next we use the equation

$$A\phi_2 - \lambda_o\phi_2 = \lambda_2\phi_o + \lambda_1\phi_1 - B\phi_1$$

which means

$$e^{-|x|} c_2 - \phi_2 = \frac{1}{2} e^{-|x|} - x^2 e^{-|x|}$$

where

$$c_2 = \int_{-\infty}^{\infty} e^{-|x|} \phi_2(x)dx \quad .$$

Thus

$$\phi_2(x) = (x^2 + c_2 - \frac{1}{2})e^{-|x|} \quad .$$

From the equation

$$0 = (\phi_o,\phi_2) + (\phi_1,\phi_1) + (\phi_2,\phi_o)$$

and the fact that ϕ_o, ϕ_1, and ϕ_2 are all real, we find

$$(\phi_o,\phi_2) = -\frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-2|x|} dx = -\frac{1}{4} \quad .$$

Hence

$$c_2 - \frac{1}{2} = -\frac{3}{4} \quad ,$$

and

$$\phi_2(x) = (x^2 - \frac{3}{4})e^{-|x|} \quad .$$

Thus we obtain

$$\lambda = 1 + \frac{1}{2} \varepsilon^2 + \ldots \quad ,$$

$$u = e^{-|x|}(1 + x\varepsilon + (x^2 - \frac{3}{4})\varepsilon^2 + \ldots) \quad .$$

Questions: Do these series converge, if we compute successively the coefficients? Do these series provide a solution of the equation

$$(A + \varepsilon B)u = \lambda u$$

with

$$\int_{-\infty}^{\infty} |u|^2 dx = 1 \quad ?$$

Unfortunately, the answers must be in the negative because our equation has no (real) point eigenvalue at all if ε is a real number not zero. Indeed if a solution $u(x) \not\equiv 0$ with $\int_{-\infty}^{\infty} |u|^2 dx < \infty$ were to exist, then

$$-c = \int_{-\infty}^{\infty} e^{-|y|} u(y)dy$$

would exist and we should have

$$-ce^{-|x|} + \varepsilon xu(x) = \lambda u(x) \quad .$$

Since ε and λ are real, $\varepsilon \neq 0$, it would then follow that

$$u(x) = \frac{ce^{-|x|}}{\varepsilon x - \lambda}$$

for $x \neq \frac{\lambda}{\varepsilon}$. The singularity at $x = \frac{\lambda}{\varepsilon}$ is a pole of order 1. Then $\int_{-\infty}^{\infty} |u|^2 dx < \infty$ is only possible if $c = 0$. But $c = 0$ would imply $u(x) \equiv 0$ which is not possible if u were an eigenfunction. As a matter of fact the spectrum of the operator $A + \varepsilon B$ is purely continuous.

The reason for the disappearance of the point eigenvalue $\lambda = 1$, present in $A + \varepsilon B$ for $\varepsilon = 0$ but vanished for $\varepsilon \neq 0$ no matter how small $|\varepsilon|$, lies in the fact that the coefficient εx in the perturbation term $\varepsilon Bu = \varepsilon xu$ is never small in the whole interval $-\infty < x < \infty$ unless $\varepsilon = 0$.

Suppose, instead of εxu, we consider the perturbation term $\varepsilon s(x)u$ where $s(x)$ is a bounded piecewise[*] continuous function in $-\infty < x < \infty$. Then we deal with the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)dy + \varepsilon s(x)u(x) = \lambda u(x) \ , \quad -\infty < x < \infty \ .$$

Again we set

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \dots \ , \quad \lambda_0 = 1 \ ,$$

$$u = \phi_0 + \varepsilon\phi_1 + \dots \ , \quad \phi_0 = e^{-|x|} \ ;$$

and we successively compute $\lambda_1, \phi_1, \dots$ as before. E.g., we find

$$\lambda_1 = \int_{-\infty}^{\infty} s(x)e^{-2|x|}dx \ , \quad \phi_1 = (s(x) - \lambda_1)e^{-|x|} \ .$$

Now we can prove that a point eigenvalue $\lambda = \lambda(\varepsilon)$ exists which can be represented as a convergent power series

$$\lambda = 1 + \varepsilon\lambda_1 + \dots$$

for small $|\varepsilon|$, however.

Assuming temporarily the existence of a point eigenvalue λ, we have for a corresponding eigenfunction u the equation

$$u(x) = \frac{ce^{-|x|}}{\lambda - \varepsilon s(x)}$$

where

---

[*] A function $f(x)$ is said to be piecewise continuous in a closed bounded interval $a \leq x \leq b$ when $f(x)$ is continuous there with the possible exception of a finite number of points x_i at which $\lim\limits_{\substack{x \to x_i \\ x < x_i}} f(x)$ and $\lim\limits_{\substack{x \to x_i \\ x > x_i}} f(x)$ exist and are finite. A function $f(x)$ is said to be piecewise continuous in an open interval $\alpha < x < \beta$ if it is piecewise continuous in every closed bounded subinterval of $\alpha < x < \beta$.

$$c = \int_{-\infty}^{\infty} e^{-|x|} u(x)dx \quad .$$

If $u(x) \not\equiv 0$ then $c \neq 0$; and we obtain the equation

$$1 = \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{\lambda - \varepsilon s(x)} dx \quad .$$

Forgetting the heuristic derivation of this equation, we shall solve it for $\lambda = \lambda(\varepsilon)$ and find $\lambda = 1 + \lambda_1 \varepsilon + \ldots$ as a power series convergent for small $|\varepsilon|$. Suppose this were done. Then for small $|\varepsilon|$ the expression $\lambda(\varepsilon) - \varepsilon s(x)$ does not vanish in $-\infty < x < \infty$, and the function

$$u = u(x;\varepsilon) = \frac{ce^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)}$$

for any constant c is such that

$$\int_{-\infty}^{\infty} |u|^2 dx < \infty \quad .$$

Furthermore, for this function u we have

$$\int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)dy + \varepsilon s(x)u(x)$$

$$= e^{-|x|} \int_{-\infty}^{\infty} \frac{ce^{-2|y|}}{\lambda(\varepsilon) - \varepsilon s(y)} dx + \frac{\varepsilon c s(x) e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)}$$

$$= ce^{-|x|} + \frac{\varepsilon c s(x) e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)}$$

$$= \lambda(\varepsilon) \frac{ce^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)} = \lambda(\varepsilon) u \quad .$$

In other words, u satisfies the eigenvalue equation with $\lambda = \lambda(\varepsilon)$. Therefore $\lambda(\varepsilon)$ is indeed a point eigenvalue.

The only gap that must be filled is the existence of a convergent power series solution

$$\lambda = \lambda(\varepsilon) = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots$$

of the functional equation

$$1 = \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{\lambda - \varepsilon s(x)}\, dx \quad .$$

We make the change of variable $\mu = \frac{\varepsilon}{\lambda}$ to obtain the equation

$$\varepsilon = \mu \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{1 - \mu s(x)}\, dx \quad .$$

If $|\mu| < \frac{1}{2s_o}$ where s_o is an upper bound of $|s(x)|$ for all x, then the integrand of the above integral can be expanded into a convergent series and termwise integration yields

$$\int_{-\infty}^{\infty} \frac{e^{-2|x|}}{1 - \mu s(x)}\, dx = 1 + \sum_{n=1}^{\infty} a_n \mu^n$$

with

$$a_n = \int_{-\infty}^{\infty} e^{-2|x|}\,[s(x)]^2\, dx \quad .$$

Since $|a_n| \le s_o^n$, the integrated power series is certainly convergent for $|\mu| < \frac{1}{2s_o}$. Hence

$$\varepsilon = \mu + \sum_{n=1}^{\infty} a_n \mu^{n+1} \quad , \quad \text{for } |\mu| < \frac{1}{2s_o} \quad ;$$

by the reversion theorem for power series,

$$\mu = \varepsilon + \varepsilon^2 \mu_1 + \dots$$

is convergent for small $|\varepsilon|$. Therefore we find

$$\lambda = \frac{\varepsilon}{\mu} = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots \quad ,$$

again convergent for small $|\varepsilon|$, which was the statement to be proved. Incidentally from $\lambda_1 = -\mu_1$ and

$$\varepsilon = (\varepsilon + \varepsilon^2\mu_1 + \dots) + a_1(\varepsilon + \mu_1\varepsilon^2 + \dots)^2 + \dots$$

we note that $\lambda_1 = -\mu_1 = a_1$, i.e.

$$\lambda_1 = \int_{-\infty}^{\infty} e^{-2|x|} \, s(x)dx$$

in agreement with our previous computation.

Thus we have shown: The equation

$$\int_{-\infty}^{\infty} e^{-|x|-|y|} \, u(y)dy + \varepsilon s(x)u(x) = \lambda u(x)$$

has a point eigenvalue $\lambda(\varepsilon)$ which is a regular analytic function of ε for small $|\varepsilon|$ and which tends to 1 as $\varepsilon \longrightarrow 0$, provided $s(x)$ is a bounded (piecewise continuous) function in $-\infty < x < \infty$. The equation has no point eigenvalue if $s(x) = x$.

We must always bear in mind that smallness of the perturbation parameter ε is not sufficient to ensure that an eigenvalue problem $(A + \varepsilon B)u = \lambda u$ has a regular analytic point eigenvalue $\lambda = \lambda_o + \varepsilon\lambda_1 + \dots$ even though the unperturbed problem $Au = \lambda u$ possesses the eigenvalue λ_o.

Thus the following two questions arise:

(1) What are criteria of "smallness" of a perturbation term εBu in general which guarantee regular analytic dependence of point eigenvalues and corresponding eigenfunctions?

(In our example such criteria were boundedness of $s(x)$ in $-\infty < x < \infty$ and smallness of $|\varepsilon|$.)

(2) If you have such a perturbation term as $\varepsilon Bu = \varepsilon s(x)u$, what is the meaning of the "approximation" computed by formally applying the perturbation method?

The eigenvalue problem just dealt with exhibits the same difficulty as that which occurs in the wave equation for the Stark effect, which was the first example in quantum mechanics treated by a perturbation calculation by E. Schrödinger[*]. An electron with charge -e moves in the electric field produced by a nucleus carrying the positive charge Ze. In addition, there is imposed an external electric field uniform in the direction of the x-axis. In suitable units the wave equation in this situation is

$$-(u_{xx}+u_{yy}+u_{zz}) - \frac{2Z}{r} u + \varepsilon x u = \lambda u \quad ,$$

where $r = (x^2+y^2+z^2)^{1/2}$ and ε is a measure of the strength of the external field. The positive integer Z and the real number ε are prescribed. A number λ_0 is said to be a point eigenvalue if for $\lambda = \lambda_0$ there is a (possibly) complex-valued solution $u \not\equiv 0$ of the differential equation such that

$$\iiint_{-\infty}^{\infty} |u(x,y,z)|^2 \, dxdydz < \infty \quad .$$

Any such solution u(x,y,z) is called an eigenfunction belonging to λ_0. For $\varepsilon = 0$ all the point eigenvalues are known. One of these eigenvalues (the lowest in fact) is $\lambda_0 = -Z^2$ with a corresponding eigenfunction $u(x,y,z) = e^{-Zr}$. We have indeed

$$-\Delta u - \frac{2Z}{r} u = -\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2Z}{r} \right) e^{-Zr} = -Z^2 u \quad .$$

"Normalizing" u we obtain the eigenfunction $\phi_0 = \frac{Z^{3/2}}{\pi^{1/2}} e^{-Zr}$.

If $\varepsilon \neq 0$ the problem is more complicated. We should like to find a solution of the form

---

[*] E. Schrödinger, Abhandlungen zur Wellenmechanik, 1926.

$$\lambda = \lambda_o + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots , \qquad \lambda_o = -Z^2$$

$$u = \phi_o + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots , \qquad \phi_o = \frac{Z^{3/2}}{\pi^{1/2}} e^{-Zr} ,$$

hoping that such expansions converge if $|\varepsilon|$ is very small. The computation of $\lambda_1, \phi_1, \dots$ is now more complicated of course. The first order approximation $\lambda_o + \varepsilon\lambda_1$ is immediately obtained, however, because

$$\lambda_1 = \iiint_{-\infty}^{\infty} x\phi_o^2 \, dxdydz = 0 ,$$

for the integrand is an odd function of x. But it is again true that our eigenvalue problem has a purely continuous spectrum if $\varepsilon \neq 0$. There is no point eigenvalue $\lambda(\varepsilon)$ for $\varepsilon \neq 0$, and the meaning of the "approximation" $\lambda(\varepsilon) \approx \lambda_o + \varepsilon\lambda_1$ is not at all clear.

The assertion that for $\varepsilon \neq 0$ there is a purely continuous spectrum is not easy to demonstrate. It amounts to proving:

If a complex-valued function $u(x,y,z) \not\equiv 0$ has continuous second derivatives and satisfies the equation

$$-\Delta u - \frac{2Z}{r} u + \varepsilon x u = \lambda u$$

for some real number $\varepsilon \neq 0$ and some complex number λ, then

$$\iiint_{-\infty}^{\infty} |u|^2 \, dxdydz = \infty .$$

It would be desirable to give a direct proof of this statement.

§2. A term of second order in the perturbation parameter cannot always be neglected in comparison with the first order term (Zeeman effect).

Usually the perturbation method in quantum mechanics is applied to operators of the form $H_0 + \varepsilon H_1$ where the perturbation parameter ε enters linearly. Even if the problem originally had the form $H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots$, one would cancel the terms of order higher than 1 and so return to $H_0 + \varepsilon H_1$. The idea is that since ε is a small parameter, terms involving ε^2 or higher powers of ε may be neglected. From our experience in the first problem we treated, we would expect that sometimes a term $\varepsilon^2 H_2$ might be not at all negligible in comparison to εH_1 as far as eigenvalues and eigenfunctions are concerned, no matter how small $|\varepsilon|$ is taken.

As an example consider the wave equation for the Zeeman effect. The equation describes the following system. An electron with charge $-e$ moves in the field produced by a nucleus, located at the origin of the x,y,z-coordinate system, bearing the positive charge Ze. In addition there is a uniform external magnetic field in the direction of the z-axis. The wave equation is

$$-(u_{xx} + u_{yy} + u_{zz}) - \frac{2Z}{r} u + \varepsilon \frac{1}{i} (xu_y - yu_x) + \varepsilon^2 \frac{x^2 + y^2}{4\mu} u = \lambda u \quad .$$

The constant ε is small if the strength of the magnetic field is small; ε is regarded as the perturbation parameter. The constant μ is the mass of an electron. The eigenvalue parameter λ is connected with the energy parameter E by $E = \frac{h^2 \lambda}{8\pi^2 \mu a^2}$, where $a = \frac{h^2}{4\pi^2 \mu e^2}$, and h is Planck's constant.

Usually the term $\varepsilon^2 \frac{x^2 + y^2}{4\mu} u$ is neglected, and the simpler equation

$$-\Delta u - \frac{2Z}{r} u + \varepsilon \frac{1}{i}(xu_y - yu_x) = \lambda u$$

is regarded as the wave equation of the Zeeman effect. For $\varepsilon = 0$, all the point eigenvalues are

$$\lambda_n = -\frac{Z^2}{n^2}\ , \qquad n = 1,2,\ldots\ ,$$

and the eigenfunctions are (in polar coordinates r, ϕ, θ) of the form

$$u_{n;\ell,m} = f_{n;\ell}(r)P_{\ell}^{m}(\cos\theta)e^{im\phi}\ , \quad \ell = 0,1,\ldots,n-1\ ,$$
$$m = 0,\pm 1,\ldots,\pm\ell\ .$$

(In the last section we had $n = 1$; $\ell = 0$, $m = 0$). Since

$$\frac{1}{i}(xu_y - yu_x) = \frac{1}{i}\frac{\partial u}{\partial\phi}\ ,$$

we find

$$\left(-\Delta - \frac{2Z}{r} + \varepsilon\,\frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)u_{n;\ell,m} = \left(-\frac{Z^2}{n^2} + \varepsilon m\right)u_{n;\ell,m}\ .$$

Thus the eigenfunctions of the unperturbed operator are also eigenfunctions of the perturbed operator belonging to the perturbed point eigenvalues

$$\lambda_n(\varepsilon) = -\frac{Z^2}{n^2} + \varepsilon m\ .$$

Thus the perturbed point eigenvalues depend analytically (even more, linearly) on ε while the perturbed eigenfunctions are independent of ε. This perturbation problem seems extremely simple. The perturbed spectrum displays a strange feature, however. Whereas the unperturbed spectrum is bounded below, the lowest eigenvalue being $-Z^2$, the spectrum of the perturbed operator is unbounded below no matter how small $|\varepsilon|$ is taken. (Note that all the numbers $-\frac{Z^2}{n^2} \pm \varepsilon(n-1)$ belong to the spectrum of the perturbed operator.) This result is unreasonable on physical grounds, and therefore it cannot be permissible to neglect the term $\varepsilon^2\,\frac{x^2+y^2}{4\mu}\,u$ in the wave equation for the Zeeman effect if one is interested in the whole spectrum of the perturbed operator and not just a single eigenvalue.

§3. Smallness of the perturbation term means smallness in comparison to the unperturbed operator.

The eigenvalue problem for the harmonic oscillator

$$- \frac{d^2u}{dx^2} + a^2x^2u = \lambda u \quad , \qquad -\infty < x < \infty \quad ,$$

with a, a given positive constant, has the point eigenvalues

$$\lambda_n = (2n+1)a \quad , \qquad n = 0,1,2,\ldots ,$$

and the corresponding (unnormalized) eigenfunctions

$$u_n = e^{-ax^2/2} H_n(\sqrt{a}\, x) \quad , \qquad n = 0,1,2,\ldots ,$$

where $H_n(x) = (-1)^n e^{x^2/2}(\frac{d}{dx})^n e^{-x^2/2}$ are the Hermite polynomials.

Adding the perturbation term εxu, we have the eigenvalue problem

$$- \frac{d^2u}{dx^2} + a^2x^2u + \varepsilon xu = \lambda u \quad , \qquad -\infty < x < \infty \quad .$$

In our previous examples we noticed that it might not be possible to regard εxu as small because x varies on the infinite interval $-\infty < x < \infty$. However, in this example we feel that the perturbation term εxu is small for small $|\varepsilon|$ in comparison to the term a^2x^2u of the unperturbed operator. In fact, the eigenvalues and eigenfunctions of the perturbation operator are

$$\lambda_n(\varepsilon) = (2n+1)a - \frac{\varepsilon^2}{4a^2} \quad ,$$

$$u_n(x;\varepsilon) = e^{-\frac{a}{2}(x + \frac{\varepsilon}{2a^2})^2} H_n(\sqrt{a}(x + \frac{\varepsilon}{2a^2})) \quad , \ n = 0,1,2,\ldots ,$$

which are regular analytic functions of ε.

A precise definition of "smallness with respect to the unperturbed operator" is one of our objectives.

§4. The character of the perturbation may depend on the sign of the perturbation parameter (String with small stiffness).

We shall consider the eigenvalue problem

$$-\frac{d^2u}{dx^2} + \varepsilon \frac{d^4u}{dx^4} = \lambda u \quad , \qquad 0 < x < 1 \quad ,$$

with certain boundary conditions on $u = u(x)$ at the endpoints $x = 0$ and $x = 1$. This problem is treated in the classical treatise of Lord Rayleigh, The Theory of Sound. The vibrations of a string with small stiffness are described by the eigenfunctions of this problem. The parameter ε is proportional to the stiffness; consequently ε is taken to be a small positive number.

First we impose the boundary conditions

$$u(0) = u''(0) = 0 \ , \quad u(1) = u''(1) = 0 \ , \quad \text{(supported ends)} \ .$$

We can immediately verify that the infinitely many numbers

$$\lambda_n = n^2\pi^2 + \varepsilon n^4\pi^4 \quad , \qquad n = 1,2,\ldots \ ,$$

are eigenvalues corresponding to the eigenfunctions

$$u_n = \sin n\pi x \qquad , \qquad n = 1,2,\ldots \ .$$

Since the functions $\sin n\pi x$, $n = 1,2,\ldots$, are complete in the interval $(0,1)$, they must be all the eigenfunctions and the numbers λ_n, $n = 1,2,\ldots$, must constitute the spectrum of our problem. Although the eigenvalues and eigenfunctions both clearly depend regular analytically on ε, there is an obvious qualitative difference when $\varepsilon \geq 0$ and when $\varepsilon < 0$. In the former case ($\varepsilon \geq 0$), the spectrum is bounded below; the lowest eigenvalue is $\lambda_1 = \pi^2 + \varepsilon\pi^4$. In the latter case ($\varepsilon < 0$), the spectrum is unbounded below, and the eigenvalues extend to $-\infty$. Of course, in the physical problem where ε measures stiffness it is physically unreasonable to permit ε to be negative.

Next we shall impose the boundary conditions

$$u(0) = u'(0) = 0\ , \quad u(1) = u'(1) = 0\ , \quad \text{(clamped ends)}\ .$$

We suppose that ε is positive. If $u(x)$ is any (complex-valued) solution of the eigenvalue equation which satisfies the boundary conditions, then it follows that

$$\int_0^1 |u'|^2 dx + \varepsilon \int_0^1 |u''|^2 dx = \lambda \int_0^1 |u|^2 dx \quad ;$$

and consequently every eigenvalue λ is positive. (We can say "eigenvalue" here rather than "point eigenvalue" because there is no continuous spectrum.) Moreover from

$$\lambda \int_0^1 |u|^2 dx \geq \int_0^1 |u'|^2 dx \quad ,$$

and the well-known inequality

$$\int_0^1 |u'|^2 dx \geq \pi^2 \int_0^1 |u|^2 dx \quad ,$$

valid for functions $u(x)$ with continuous first derivatives in $(0,1)$ and with $u(0) = u(1) = 0$, we can conclude that

$$\lambda \geq \pi^2$$

for all the eigenvalues of our problem.

We look for the eigenvalues. The function $e^{\rho x}$ is a solution of our differential equation provided

$$-\rho^2 + \varepsilon\rho^4 = \lambda \quad .$$

Thus we find

$$\rho^2 = \frac{1 \pm a}{2\varepsilon} \quad , \quad \text{where } a = (1 + 4\lambda\varepsilon)^{1/2} \quad .$$

Notice that since ε was assumed positive and λ will be positive if it is to be an eigenvalue, we have $\alpha > 1$. Thus four linearly independent solutions of our differential equation are

$$e^{rx} \ , \quad e^{isx} \ , \quad e^{-rx} \ , \quad e^{-isx} \ ,$$

where r and s are the positive numbers satisfying

$$r^2 = \frac{\alpha + 1}{2\varepsilon} \ , \quad s^2 = \frac{\alpha - 1}{2\varepsilon} \ .$$

Four other linearly independent solutions are

$$\sin sx \ , \quad \cos sx \ , \quad \sinh rx \ , \quad \cosh rx \ ,$$

Every solution u(x) of the differential equation can be written as

$$u(x) = c_1 \sin sx + c_2 \cos sx + c_3 \sinh rx + c_4 \cosh rx \ .$$

To satisfy the boundary conditions we must require:

$$\begin{aligned}
u(0) &= \qquad c_2 \qquad + c_4 = 0 \ , \\
u'(0) &= c_1 s \qquad + c_3 r \qquad = 0 \ , \\
u(1) &= c_1 \sin s + c_2 \cos s + c_3 \sinh r + c_4 \cosh r = 0 \ , \\
u'(1) &= c_1 s \cos s - c_2 s \sin s + c_3 r \cosh r + c_4 r \sinh r = 0 \ .
\end{aligned}$$

This system of four linear equations in the four unknowns c_1, c_2, c_3, c_4 has a non-trivial solution if and only if its determinant Δ is zero. A computation yields

$$\Delta = 2rs - 2rs \cosh r \cos s + (r^2 - s^2) \sinh r \sin s \ .$$

The eigenvalues of our problem are precisely the positive zeros of Δ, or of

$$\delta = \frac{2}{r^2 e^r}\,\Delta = 4\frac{s}{r}\,e^{-r} - 2\frac{s}{r}(1 + e^{-2r})\cos s + \left(1 - \frac{s^2}{r^2}\right)(1 - e^{-2r})\sin s \quad .$$

Now let us assume that there is an eigenvalue $\lambda(\varepsilon)$ defined for all small positive ε, and further let us assume that as ε tends to zero through positive numbers that $\lambda(\varepsilon)$ tends to a limit $\overset{\circ}{\lambda}$. Since as we have already seen $\lambda(\varepsilon) \geq \pi^2$, it follows that $\overset{\circ}{\lambda} \geq \pi^2$. For sufficiently small ε, we have

$$\alpha = 1 + 2\lambda(\varepsilon)\varepsilon + o(\lambda(\varepsilon)\varepsilon) \quad ;$$

and hence

$$s = [\lambda(\varepsilon)]^{1/2}(1 + o(1))^{1/2} \quad ,$$

and

$$\lim_{\varepsilon \to 0+} s = \overset{\circ}{\lambda}{}^{1/2} = s_o \quad ;$$

and

$$\lim_{\varepsilon \to 0+} r = +\infty \quad .$$

It then follows from an inspection of the equation $\delta = 0$ that $\sin s_o = 0$, which implies that $s_o = n\pi$, $n = 1,2,\ldots$. Therefore we have

$$\overset{\circ}{\lambda} = \overset{\circ}{\lambda}_n = n^2\pi^2 \quad , \qquad n = 1,2,\ldots,$$

which are just the eigenvalues of the problem

$$-u'' = \lambda u \quad , \qquad 0 < x < 1$$

with the boundary conditions

$$u(0) = u(1) = 0 \quad .$$

We have proved that the perturbed eigenvalues $\lambda(\varepsilon)$ for $\varepsilon > 0$ can have as limits as $\varepsilon \to 0+$ only the spectrum of the eigenvalue problem for the string (without stiffness) with fixed endpoints.

Certainly we should not be surprised that the limits $\overset{\circ}{\lambda}$ are connected with the unperturbed equation $-u'' = \lambda u$. We may ask, however, why the particular boundary conditions $u(0) = u(1) = 0$ are distinguished. It is a priori just as plausible that from the boundary conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0$$

of the perturbed problem, some other pair of boundary conditions, e.g. $u(0) = u'(1) = 0$, might be singled out for the unperturbed equation. In that case the spectrum of the unperturbed problem would consist of the numbers $(2n-1)^2 \frac{\pi^2}{4}$, $n = 1,2,\ldots$. Note also that when we imposed the boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0 \quad ,$$

we obtained exactly the same limits $\overset{\circ}{\lambda} = n^2\pi^2$, $n = 1,2,\ldots$, corresponding again to the boundary conditions $u(0) = u(1) = 0$ for the unperturbed equation.

We leave these questions aside to study the character of the dependence of an eigenvalue $\lambda(\varepsilon)$ of the perturbed problem (clamped ends) on ε as $\lambda(\varepsilon)$ approaches one of the numers $n^2\pi^2$, $n = 1,2,\ldots$, as ε tends to zero through positive values. Since ε and $\lambda = \lambda(\varepsilon)$ are positive we can define new variables η, ν, and κ by the relations

$$\varepsilon = \eta^2 \ , \quad \lambda = \nu^2 \ , \quad \nu = n\pi + \kappa \ ;$$

where n is a fixed positive integer. We can suppose that $|\kappa| < C$ where C is some constant; we shall only be interested in small κ. Then we obtain

$$a = (1+4\nu^2\eta^2)^{1/2} = 1+2\nu^2\eta^2 - 2(\nu^2\eta^2)^2 + \dots \quad ,$$

$$r = \left(\frac{(1+4\nu^2\eta^2)^{1/2}+1}{2\eta^2}\right)^{1/2} = \frac{1}{\eta} + \frac{1}{2}\nu^2\eta^2 + \dots \quad ,$$

$$s = \left(\frac{(1+4\nu^2\eta^2)^{1/2}-1}{2\eta^2}\right)^{1/2} = \nu(1-\frac{1}{2}\nu^2\eta^2 + \dots) \quad .$$

We denote by $f_1(\kappa,\eta)$, $f_2(\kappa,\eta)$,... power series in κ, η which are convergent in a neighborhood of $\kappa = \eta = 0$ and which vanish at $\kappa = \eta = 0$. Then we have

$$r = \frac{1}{\eta} + f_1(\kappa,\eta) = \frac{1+\eta f_1(\kappa,\eta)}{\eta} \quad ,$$

$$s = n\pi + \kappa + \eta f_2(\kappa,\eta) \quad ,$$

$$\frac{s}{r} = \eta(n\pi + f_3(\kappa,\eta)) \quad ,$$

$$e^{-r} = e^{-1/\eta}(1+f_4(\kappa,\eta)) \quad .$$

If we set $\theta = e^{-1/\eta} = \exp(-\varepsilon^{-1/2})$, then the function $\delta = \frac{2\Delta}{r^2e^r}$ is a power series in κ, η, θ convergent in a neighborhood of $\kappa = \eta = \theta = 0$. (As a matter of fact δ is merely a polynomial of degree 2 in θ.) Thus δ is of the form

$$\delta = -(-1)^n 2n\pi\eta + (-1)^n\kappa + P(\kappa;\eta,\theta)$$

where P is a power series in κ, η, θ, convergent in a neighborhood of $\kappa = \eta = \theta = 0$, containing no terms of order 0 or 1. Our aim is to solve the equation

$$\delta = \delta(\kappa;\eta,\theta) = 0$$

for κ. We shall show that

$$\kappa = 2n\pi\eta + Q(\eta,\theta)$$

where $Q(\eta,\theta)$ is a power series in η, θ, convergent in a neighborhood of $\eta = \theta = 0$, and containing no linear term.

We use the Weierstrass Preparation Theorem[*]:

Suppose that $F(z;x_1,\dots,x_h)$ is a power series in the $h+1$ variables $z;x_1,\dots,x_h$ convergent in a neighborhood of $(0;0,\dots,0)$, and suppose that

$$F(z;0,\dots,0) = c_0 z^k + c_1 z^{k+1} + \dots , \qquad c_0 \neq 0 .$$

Then

$$F(z;x_1,\dots,x_h) = [c_0 z^k + p_1(x_1,\dots,x_h) z^{k-1} + \dots + p_k(x_1,\dots,x_h)]\, E(z;x_1,\dots,x_h) ,$$

where $p_1,\dots,p_k$ are power series in $x_1,\dots,x_k$ convergent in a neighborhood of $(0,\dots,0)$ and vanishing at $(0,\dots,0)$ and where E is a power series convergent in a neighborhood of $(0;0,\dots,0)$ and $E(0;0,\dots,0) = 1$.

In our application we have $k = 1$, i.e.

$$\delta(\kappa;0,0) = (-1)^n \kappa + \dots$$

so that

$$\delta(\kappa;\eta,\theta) = [(-1)^n \kappa + p_1(\eta,\theta)]E(\kappa;\eta,\theta) .$$

Evidently we find

$$p_1(\eta,\theta) = (-1)^n\, 2n\pi\eta + P_1(\eta,\theta) ,$$

where $p_1(\eta,\theta)$ begins with quadratic terms. For $|\kappa|$ small enough, those κ satisfying $\delta(\kappa;\eta,\theta) = 0$ are of the form

$$\kappa = (-1)^n p_1(\eta;\theta) = 2n\pi\eta + Q(\eta,\theta)$$

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[*] For a proof, see C. L. Siegel, *Analytic Functions of Several Complex Variables*, Lectures derlivered at the Institute for Advanced Study, (1948-49), p. 3.

where $Q(\eta,\theta)$ is a power series in η, θ convergent in a neighborhood of $\eta = 0$, $\theta = 0$ and beginning with quadratic terms. We conclude that

$$\nu = n\pi + 2n\pi\eta + Q(\eta,\theta)$$

or

$$\lambda(\varepsilon) = n^2\pi^2 + 4n^2\pi^2\varepsilon^{1/2} + R(\varepsilon^{1/2}, e^{-\varepsilon^{-1/2}})$$

for an eigenvalue of the perturbed problem which tends to $n^2\pi^2$ as ε tends to zero through positive values. Here $R(\eta,\theta)$ is a power series in η, θ convergent in a neighborhood of $\eta = 0$, $\theta = 0$, beginning with quadratic terms, and consequently the nature of the dependence of $\lambda(\varepsilon)$ on ε is completely exposed. Note that $\lambda(\varepsilon)$ is not a regular analytic function of ε for small $|\varepsilon|$.

Further analysis would reveal that for negative ε, $\lambda(\varepsilon)$ behaves quite differently as $\varepsilon \longrightarrow 0-$ from its behavior for positive ε.

§5. Non-analytic perturbation.

In perturbation theory the approximating problems usually can be easily treated whereas the original problem is more difficult. There are cases where the opposite is nearly true, however. This situation occurs when it is hard to understand the physical meaning of the mathematical problem or some part of it. Then one substitutes approximating problems which may be more difficult mathematically but whose physical meaning is clear.

For example consider the wave equation for a free particle in quantum mechanics

$$-(u_{xx} + u_{yy} + u_{zz}) = \lambda u \quad .$$

To simplify the mathematical techniques used, we restrict the particle to the unit sphere $x^2 + y^2 + z^2 < 1$ imposing the boundary condition $u = 0$ for $x^2 + y^2 + z^2 = 1$. A number λ is

said to be an eigenvalue, (as there is no continuous spectrum we say "eigenvalue" instead of "point eigenvalue"), if there is a solution $u \not\equiv 0$ of the differential equation which satisfies the boundary condition. Since the physical interpretation of $|u(x,y,z)|^2$ is that of a probability density, we further require

$$\iiint\limits_{x^2+y^2+z^2\leq 1} |u(x,y,z)|^2 \, dxdydz = 1 \quad .$$

Now we seek eigenfunctions which depend only on $r = (x^2+y^2+z^2)^{1/2}$, i.e. we try $u = f(r)$. We obtain for $u = f(r)$ the ordinary differential equation

$$-\frac{d^2f}{dr^2} - \frac{2}{r}\frac{df}{dr} = \lambda f \quad , \qquad 0 < r < 1 \quad ,$$

with the auxiliary conditions

$$f(1) = 0$$

and

$$4\pi \int_0^1 |f|^2 \, r^2 dr = 1 \quad .$$

If we set $g = \sqrt{4\pi}\, f(r)r$, the differential equation simplifies to

$$-\frac{d^2g}{dr^2} = \lambda g \quad , \qquad 0 < r < 1 \quad ,$$

with the auxiliary conditions

$$g(1) = 0$$

and

$$\int_0^1 |g|^2 \, dr = 1 \quad .$$

If we impose the additional boundary condition

$$g(0) = 0 \quad ,$$

then we obtain the eigenvalues

$$\lambda_n = n^2\pi^2 \quad , \qquad n = 1,2,\dots ,$$

and the corresponding eigenfunctions

$$g_n(r) = \sqrt{2}\,\sin n\pi r \quad , \qquad n = 1,2,\dots .$$

Thus we find the eigenfunctions

$$u_n = \frac{1}{\sqrt{2\pi}}\,\frac{\sin n\pi r}{r} \quad , \qquad n = 1,2,\dots$$

for the free particle in the unit sphere. These eigenfunctions are certainly continuous in the unit sphere; indeed they are analytic for $0 \leq r \leq 1$. However, we could replace the boundary condition $g(0) = 0$ by any of the conditions

$$g'(0) + \sigma g(0) = 0$$

where σ is a real number. Then we would obtain eigenfunctions $g_n(r)$ with $\int_0^1 |g_n|^2\,dr = 1$ which are continuous (even regular analytic) in $0 \leq r \leq 1$, but which do not vanish at $r = 0$. Consequently the eigenfunctions

$$u_n(x,y,z) = \frac{g_n(r)}{\sqrt{4\pi}\,r}$$

are not continuous at the origin, though they are continuous (even regular analytic) for $0 < x^2 + y^2 + z^2 \leq 1$, and they satisfy $\iiint_{x^2+y^2+z^2\leq 1} |u|^2\,dxdydz = 1$. Since the quantity $|u|^2$

in Schrödinger's theory represents a probability density, the fact that $|u_n|^2$ becomes infinite at a point is not incompatible with its physical interpretation. Therefore one would believe that these alternate boundary conditions are not only mathematically reasonable but also have a physical meaning for the problem in quantum mechanics.

Again we consider a particle moving in the sphere $x^2 + y^2 + z^2 < 1$, but now we imagine it moving in a field derived from a potential V. We assume that V is zero outside some small sphere $x^2 + y^2 + z^2 \leq \varepsilon^2 < 1$ and the V is some negative constant, say $V = -V_o$, in the small sphere. We demand that V be independent of x, y, z, but V may depend on ε. Then we consider the eigenvalue problem

$$-(u_{xx} + u_{yy} + u_{zz}) + Vu = \lambda u \quad , \qquad x^2 + y^2 + z^2 < 1$$

with the boundary condition

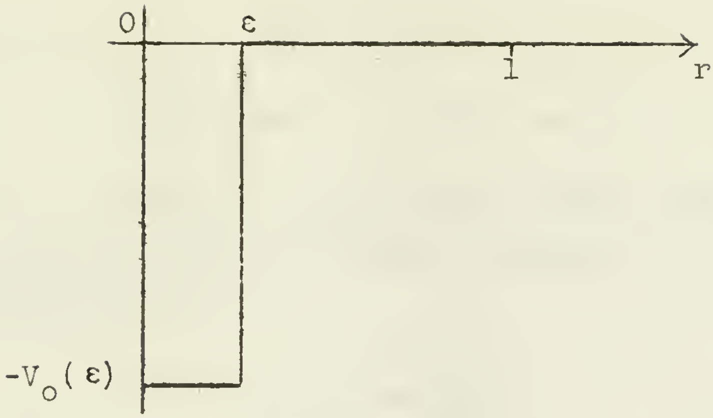
$$u = 0 \qquad\qquad \text{for} \quad x^2 + y^2 + z^2 = 1 \quad .$$

Again we seek eigenfunctions which depend only on r, say $u = f(r)$, and we set $g = \sqrt{4\pi}\, f(r)r$. We obtain the differential equation

$$-\frac{d^2g}{dr^2} + q(r)g = \lambda g \quad , \qquad 0 < g < 1 \quad ,$$

and the boundary condition $g(1) = 0$. The function $q(r)$ is defined as

$$q(r) = \begin{cases} -V_o & \text{for } 0 < r \leq \varepsilon \quad , \\ 0 & \text{for } \varepsilon \leq r < 1 \quad . \end{cases}$$



Graph of potential q(r)

We impose the additional boundary condition $g(0) = 0$. Then we find the eigenfunctions $u = \dfrac{g_n(r)}{\sqrt{4\pi}\, r}$ of the original problem which are continuous at the origin, and evidently represent bound states of the physical problem.

We now wish to choose $V_o = V_o(\varepsilon)$ in such a way that the eigenvalue problem for g approximates the eigenvalue problem

$$-\frac{d^2\gamma}{dr^2} = \lambda\gamma \quad , \qquad 0 < r < 1 \quad ,$$

and

$$\gamma'(0) + \sigma\gamma(0) = 0 \quad , \qquad \gamma(1) = 0 \quad .$$

We first make a preliminary heuristic investigation which will indicate the appropriate choice of $V(\varepsilon)$. Suppose that λ is a particular eigenvalue with $\gamma(r)$ a corresponding eigenfunction; and suppose $\lambda(\varepsilon)$ and $g(r)$ are the eigenvalue and eigenfunction, respectively, of the perturbed problem

$$-\frac{d^2g}{dr^2} + q(r)g = \lambda(\varepsilon)g \quad , \qquad 0 < r < 1 \quad ,$$

and

$$g(0) = g(1) = 0 \quad ,$$

such that

$$\lambda(\varepsilon) \longrightarrow \lambda \quad , \qquad g(r) \longrightarrow \gamma(r)$$

as $\varepsilon \longrightarrow 0$. Since in $\varepsilon \leq r < 1$ the differential equations for g and γ are nearly the same and $g(1) = \gamma(1) = 0$, we should expect that in $\varepsilon \leq r < 1$

$$\frac{g'(r)}{g(r)} \approx \frac{\gamma'(r)}{\gamma(r)} \quad .$$

Furthermore since $\gamma'(0) + \sigma\gamma(0) = 0$, we should expect $g'(\varepsilon) + \sigma g(\varepsilon) \approx \gamma'(\varepsilon) + \sigma\gamma(\varepsilon) \approx 0$. Now in $0 < r < \varepsilon$, we find

$$g(r) = a \sin(\sqrt{V_o + \lambda(\varepsilon)}\; r) \quad ,$$

where a is some constant. It seems reasonable that $V_o(\varepsilon) \rightarrow \infty$ as $\varepsilon \longrightarrow 0$. Since $\lambda(\varepsilon) \longrightarrow \lambda$ as $\varepsilon \longrightarrow 0$, we can write

$$\sqrt{V_o + \lambda(\varepsilon)} \approx \sqrt{V_o} \quad .$$

Then we obtain

$$\sqrt{V_o} \cos \sqrt{V_o}\, \varepsilon + \sigma \sin \sqrt{V_o}\, \varepsilon \approx 0 \quad ,$$

and as $\varepsilon \longrightarrow 0$, we must require that $\sqrt{V_o}\,\varepsilon$ tend to a zero x_n of cos x, say

$$\sqrt{V_o}\,\varepsilon \longrightarrow x_n = (n - \tfrac{1}{2})\pi \quad ,$$

where n is some positive integer. Let us define $\rho = \rho(\varepsilon)$ by the relation

$$\sqrt{V_o}\,\varepsilon = x_n + \rho \quad ;$$

$\lim\limits_{\varepsilon \to 0} \rho = 0$. Thus we obtain

$$-\sqrt{V_o} \sin x_n \sin \rho + \sigma \sin x_n \cos \rho \approx 0 \quad ,$$

and hence

$$\sqrt{V_o}\,\rho \longrightarrow \sigma$$

as $\varepsilon \longrightarrow 0$. Thus we have as $\varepsilon \longrightarrow 0$

$$\sqrt{V_o}\,\rho = \frac{x_n + \rho}{\varepsilon}\,\rho \longrightarrow \sigma \quad ,$$

and consequently

$$\rho \approx \frac{\sigma\varepsilon}{x_n} \quad .$$

Therefore we obtain

$$\sqrt{V_o} \approx \frac{x_n}{\varepsilon} + \frac{\sigma}{x_n} \quad ,$$

and

$$V_o \approx \frac{x_n^2}{\varepsilon^2} + \frac{2\sigma}{\varepsilon} \quad .$$

We begin anew by defining

$$V_o = \frac{(n-\frac{1}{2})^2\pi^2}{\varepsilon^2} + \frac{2\sigma}{\varepsilon} \quad ,$$

and we indicate how you might try to prove the result rigorously. First we find the exact equation for the perturbed eigenvalue λ. We have

$$g(r) = \begin{cases} \sin\sqrt{V_o + \lambda(\varepsilon)}\; r & \text{for } 0 < r < \varepsilon \quad , \\ c\,\sin\sqrt{\lambda(\varepsilon)}\;(1-r) & \text{for } \varepsilon \leq r < 1 \quad , \end{cases}$$

where c is some number. For $r = \varepsilon$ we have

$$\sin\sqrt{V_o + \lambda(\varepsilon)}\;\varepsilon = c\,\sin\sqrt{\lambda(\varepsilon)}\;(1-\varepsilon) \quad ,$$

$$\sqrt{V_o + \lambda(\varepsilon)}\,\cos\sqrt{V_o + \lambda(\varepsilon)}\;\varepsilon = -c\,\sqrt{\lambda(\varepsilon)}\,\cos\sqrt{\lambda(\varepsilon)}\;(1-\varepsilon) \quad ,$$

and hence

$$\sqrt{\lambda(\varepsilon)}\cos[\sqrt{\lambda(\varepsilon)}(1-\varepsilon)]\sin[\sqrt{V_0+\lambda(\varepsilon)}\,\varepsilon]$$
$$+\sqrt{V_0+\lambda(\varepsilon)}\sin[\sqrt{\lambda(\varepsilon)}(1-\varepsilon)]\cos[\sqrt{V_0+\lambda(\varepsilon)}\,\varepsilon]=0 \quad .$$

If $V_0 = \frac{x_n^2}{\varepsilon^2} + \frac{2\sigma}{\varepsilon}$, then as $\varepsilon \to 0$ it is clear that this equation goes into

$$\sqrt{\lambda}\cos\sqrt{\lambda} - \sigma\sin\sqrt{\lambda} = 0$$

which is precisely the equation for the eigenvalue of the unperturbed problem. We shall not complete the proof that $\lambda(\varepsilon) \to \lambda$ as $\varepsilon \to 0$.

Writing $A\gamma = \lambda\gamma$ for the unperturbed problem and $A_\varepsilon g = \lambda g$ for the perturbed one, we notice a significant difference between this and our previous examples of the perturbation method. In this case we cannot write A_ε as $A + \varepsilon B$ or even as $A + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots$. We know that if we set

$$V_0 = \frac{(n-\frac{1}{2})^2\pi^2 + 2\varepsilon\sigma}{\varepsilon^2} \quad , \qquad n = 1,2,\ldots ,$$

the equation for the perturbed eigenvalues goes into that for the unperturbed ones as $\varepsilon \to 0$. It is one of our objectives to clarify the sense in which the operator A_ε approximates A even though A_ε does not depend analytically on ε.

Finally let us remark that the case $n = 1$ above is different from $n > 1$ in that the spectra of A_ε for $n = 1$ are bounded below uniformly for $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is some positive number, whereas this is not true for $n = 2,3,\ldots$. It is easy to prove the second part of this assertion; we omit the proof. We shall prove the first part.

We shall prove that there is a positive number ε_0 such that the spectra of A_ε for $0 < \varepsilon < \varepsilon_0$ all have the same lower bound. We denote by $\mathcal{D}$ the set of all complex-valued functions $g(r)$ defined for $0 \leq r \leq 1$ having the following properties:

(1) g and g' continuous and g'' piecewise continuous in $0 \leq r \leq 1$,

(2) $g(0) = g(1) = 0$.

This set $\mathcal{D}$ is called the domain of the operator A_ε which is defined as

$$A_\varepsilon g = -\frac{d^2g}{dr^2} + g(r)g$$

where

$$q(r) = \begin{cases} -\left(\frac{\pi}{2\varepsilon}\right)^2 - \frac{2\sigma}{\varepsilon} & \text{for } 0 < r < \varepsilon\ , \\ 0 & \text{for } \varepsilon \leq r < 1\ . \end{cases}$$

The spectrum of A_ε on $\mathcal{D}$ is bounded below if there exists a real number a such that

$$(g, A_\varepsilon g) \geq a(g,g)$$

for all functions g in $\mathcal{D}$. We shall prove that we may take $a = 0$ for all $0 < \varepsilon < \varepsilon_0$ when $\sigma \leq 1$, and $a = -\sigma^2$ for all $0 < \varepsilon < \varepsilon_0$ when $\sigma > 1$; ε_0 is some positive number less than 1.

In any event we have

$$(g, A_\varepsilon g) = \int_0^1 |g'|^2\, dr - \left(\left(\frac{\pi}{2\varepsilon}\right)^2 + \frac{2\sigma}{\varepsilon}\right)\int_0^\varepsilon |g|^2\, dr \quad .$$

We first suppose that $\sigma < 1$. We shall now construct a real-valued function $w = w_\varepsilon(r)$ defined for $0 \leq r \leq 1$ with the following properties:

(1) w and w' continuous and w'' piecewise continuous,

(2) $w > 0$ in $0 < r \leq 1$, $w_r(0) > 0$,

(3) $A_\varepsilon w \geq 0$.

Let us denote by ν the positive square root

$$\nu = \left(\frac{\pi^2}{4\varepsilon^2} + \frac{2\sigma}{\varepsilon} \right)^{\frac{1}{2}} \quad .$$

Let

$$w_\varepsilon(r) = \begin{cases} \sin \nu r & \text{for} \quad 0 \leq r < \varepsilon \quad , \\ (\nu \cos \nu\varepsilon)(r-\varepsilon) + \sin \nu\varepsilon & \text{for} \quad \varepsilon \leq r \leq 1 \quad . \end{cases}$$

It is evident that w thus defined has property (1), and that $A_\varepsilon w = 0$. We must only check condition (2). Now as $\varepsilon \longrightarrow 0+$ we have

$$\nu = \frac{\pi}{2\varepsilon} + \frac{2\sigma}{\pi} + o(1) \quad ,$$

$$\sin \nu\varepsilon = 1 + o(1) \quad ,$$

$$\nu \cos \nu\varepsilon = -\sigma + o(1) \quad .$$

To see that $w > 0$ it is enough to make sure that $w_\varepsilon(1) = (\nu \cos \nu\varepsilon)(1-\varepsilon) + \sin \nu\varepsilon = 1 - \sigma + o(1)$ is positive. This is certainly true for ε small enough, say for $0 < \varepsilon < \varepsilon_0 < 1$.

Now define $f(r)$ by the equation

$$g(r) = f(r)w(r) \quad ;$$

then for $g(r)$ real-valued, we have

$$(g, A_\varepsilon g) = \lim_{\delta \longrightarrow 0} \left(\int_\delta^1 (f'w + fw')^2 dr - \left(\frac{\pi^2}{4\varepsilon^2} + \frac{2\sigma}{\varepsilon} \right) \int_\delta^\varepsilon f^2 w^2 dr \right) \quad .$$

Integration by parts yields

$$(g,A_\varepsilon g) = \lim_{\delta \to 0} \left((f^2ww')\Big|_\delta^1 + \int_\delta^1 (f'^2w^2 + f^2w^2(A_\varepsilon w))\,dr \right) .$$

Now f^2ww' vanishes at 1, and $\lim_{\delta \to 0} f^2ww'\Big|_\delta = \lim_{\delta \to 0} \frac{g^2w'}{w} = 0$ because $g(\delta) = g'(0)\delta + o(\delta)$ and $w = w'(0)\delta + o(\delta)$ and $w'(0) \neq 0$. It is then evident that

$$(g,A_\varepsilon g) \geq 0$$

for $0 < \varepsilon < \varepsilon_o$.

In case $\sigma \geq 1$, the above argument fails because w as defined is not positive. Indeed it is no longer true that $(g,A_\varepsilon g) \geq 0$ for small ε.

Let us redefine ν as

$$\nu = \left(\frac{\pi^2}{4\varepsilon^2} + \frac{2\sigma}{\varepsilon} - \sigma^2 \right)^{\frac{1}{2}}$$

for ε so small that the quantity in the parentheses is positive. We find as $\varepsilon \to 0$ that

$$\nu\varepsilon = \frac{\pi}{2} + \frac{2\sigma}{\pi}\varepsilon - \frac{\sigma^2}{\pi}\left(1 + \frac{4}{\pi^2}\right)\varepsilon^2 + O(\varepsilon^3) \quad ,$$

$$\sin \nu\varepsilon = 1 + O(\varepsilon^2)$$

$$\nu \cos \nu\varepsilon = - \sigma + \sigma^2\left(-\frac{2}{\pi^2} + \frac{1}{2}\right)\varepsilon + O(\varepsilon^2) \quad .$$

Now we define $w = w_\varepsilon(r)$ as

$$w_\varepsilon(r) = \begin{cases} \sin \nu r & \text{for } 0 \leq r < \varepsilon \ , \\ (\nu \cos \nu\varepsilon)(r-\varepsilon) + \sin \nu\varepsilon & \text{for } \varepsilon \leq r < \alpha \ , \\ (\sin \nu\varepsilon)\, e^{-\sigma(r-\beta)} & \text{for } \alpha \leq r < 1 \ . \end{cases}$$

where α and β are numbers chosen so that $\varepsilon \leq \alpha \leq 1$ and $w_\varepsilon(r)$ has a piecewise continuous second derivative in $(0,1)$. It is easy to see that this is possible because the functions $(\nu \cos \nu\varepsilon)(r-\varepsilon) + \sin \nu\varepsilon$ and $(\sin \nu\varepsilon)\, e^{-\sigma(r-\varepsilon)}$ are equal at $r = \varepsilon$, and the (always positive) exponential function at $r = \varepsilon$ has a steeper slope than the linear function, provided ε is small enough. Finally notice that $(A_\varepsilon + \sigma^2)w \geq 0$ for $0 < r < 1$; in fact, the equality holds except for $\varepsilon < r < \alpha$. Therefore the same proof as before yields

$$(g,(A_\varepsilon + \sigma^2)g) \geq 0$$

or

$$(g,A_\varepsilon g) \geq -\sigma^2(g,g)$$

for $0 < \varepsilon < \varepsilon_0$.

Merely as a technical convenience, we made the unnecessary assumption that $g(r)$ is real-valued. We shall not bother about removing it, however.

Chapter I. Finite Dimensional Space

§1. If the operator is a convergent power series in the perturbation parameter, then so are the eigenvalues and eigenvectors.

We shall consider the eigenvalue problem

$$\sum_{k=1}^{n} a_{ik}u_k = \lambda u_i \quad , \qquad i = 1,\ldots,n \quad ,$$

where the (a_{ik}) are given complex numbers; the problem is to find λ and corresponding $u_1,\ldots,u_n$. A complex number λ for which there exists a solution $u = (u_1,\ldots,u_n)$ of the n equations, with $u \neq (0,\ldots,0)$, is said to be an eigenvalue; the solution u is called an eigenvector (or eigenelement) belonging to λ. The eigenvector u is a vector in the n-dimensional vector space $\mathfrak{H}^n$, which consists of n-tuples of complex numbers. We introduce the usual norm

$$\|u\| = (|u_1|^2 + \ldots + |u_n|^2)^{\frac{1}{2}}$$

and the associated inner product

$$(v,u) = \bar{v}_1 u_1 + \ldots + \bar{v}_n u_n \quad ,$$

where $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$ are any vectors in $\mathfrak{H}^n$; we have

$$\|u\| = (u,u)^{\frac{1}{2}} \quad .$$

Frequently we demand that the eigenelements be normalized, i.e. $\|u\| = 1$.

We denote by A the operator (or transformation) which transforms an element $u = (u_1,\ldots,u_n)$ of $\mathfrak{H}^n$ into the element $v = (v_1,\ldots,v_n)$ in $\mathfrak{H}^n$ by the rule

$$v_i = \sum_{k=1}^{n} a_{ik}u_k \quad , \qquad i = 1,\ldots,n \quad .$$

Then our eigenvalue problem assumes the form

$$Au = \lambda u \quad .$$

If the operator A is Hermitian, i.e. $a_{ik} = \overline{a_{ki}}$ or equivalently $(v,Au) = (Av,u)$ for all elements u, v in $\mathfrak{H}^n$, then all the eigenvalues are real numbers. In fact there exist n linearly independent eigenelements $\phi_1,\ldots,\phi_n$ corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_n$, respectively, i.e. $A\phi_\nu = \lambda_\nu\phi_\nu$, $\nu = 1,\ldots,n$; and we can suppose the eigenelements to be orthonormalized,

$$(\phi_\nu,\phi_\mu) = \delta_{\nu\mu} = \begin{cases} 0 & \text{for} \quad \nu \neq \mu \quad , \\ 1 & \text{for} \quad \nu = \mu \quad . \end{cases}$$

Suppose now that the coefficients a_{ik} are power series in ε all convergent in a neighborhood of $\varepsilon = 0$. Furthermore suppose that the matrix $(a_{ik}(\varepsilon))$ is still Hermitian for real ε within the neighborhood of convergence, i.e. $a_{ik}(\varepsilon) = \overline{a_{ki}(\varepsilon)}$ for real ε. The eigenvalues and eigenvectors of the perturbed eigenvalue problem depend on ε. Our question is: Are the perturbed eigenvalues and eigenvectors regular analytic functions of ε for small $|\varepsilon|$?

In general to solve a problem involving both eigenvalues and eigenvectors, it is necessary to treat both simultaneously. There are some simple cases when they can be treated separately, however. The eigenvalue problem in a finite dimensional space is an example, for here the eigenvalues are just the roots of the secular equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad .$$

This determinant is evidently a polynomial of degree n in λ, say

$$\lambda_n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

where $c_1, \ldots, c_n$ are power series in ε converging for small $|\varepsilon|$. Our question about the eigenvalues reduces to asking whether or not the zeros of such a polynomial are regular analytic functions of ε for small $|\varepsilon|$. In general the answer is no; a counterexample is $\lambda^2 + \varepsilon$. What is true (Puiseux's expansion) is that if $\lambda = \lambda(0)$ is a zero for $\varepsilon = 0$, then the zero $\lambda(\varepsilon)$ can be written as a convergent (for small $|\varepsilon|$) power series in $\varepsilon^{\frac{1}{h}}$ where h is the multiplicity of $\lambda = \lambda(0)$.

Of course we have not yet used the special properties of our polynomial. In particular we know that all the zeros are real when ε is real. This implies that in Puiseux's expansion of $\lambda(\varepsilon)$ only terms with integral powers of ε can appear. To prove this, we write

$$\lambda(\varepsilon) = \lambda + b_1\varepsilon^{\frac{1}{h}} + b_2\varepsilon^{\frac{1}{h}} + \cdots \quad .$$

This series is convergent for small $|\varepsilon|$. Let b_n denote the first coefficient not zero, i.e.

$$b_1 = \cdots = b_{n-1} = 0 \quad , \qquad b_n \neq 0 \quad .$$

then letting $\varepsilon \longrightarrow 0$ through real positive values, we see that

$$b_n = \lim_{\varepsilon \to 0+} \frac{\lambda(\varepsilon) - \lambda}{\varepsilon^{\frac{n}{h}}}$$

is real, because $\lambda(\varepsilon)$ is real for ε real. On the other hand letting $\varepsilon \longrightarrow 0$ through negative real numbers we note that

$$(-1)^{\frac{n}{h}} b_n = \lim_{\varepsilon \to 0-} \frac{\lambda(\varepsilon) - \lambda}{(-\varepsilon)^{\frac{n}{h}}}$$

is also real. Hence $(-1)^{\frac{n}{h}}$ is a real number, and therefore n must be a multiple of h. We can continue this argument to show that only integral powers of ε can have non-zero coefficients.

Thus we have proved:

The eigenvalues of a Hermitian matrix $(a_{ik}(\varepsilon))$ (i.e. $a_{ik}(\varepsilon) = \overline{a_{ki}(\varepsilon)}$ for real ε with $|\varepsilon|$ small) whose elements are convergent power series for small $|\varepsilon|$ can be considered as power series in ε convergent for small $|\varepsilon|$.

Remark: We use the phrase "can be considered" to indicate that a suitable arrangement of the eigenvalues $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$ must be made for the above assertion to be true. Consider for example the Hermitian matrix

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

whose eigenvalues are $\lambda_1(\varepsilon) = \varepsilon$ and $\lambda_2 = -\varepsilon$, evidently convergent power series. But if one were to arrange them in order of magnitude, then $\lambda^1(\varepsilon) = -|\varepsilon|$ and $\lambda^2(\varepsilon) = |\varepsilon|$ which are not regular analytic functions of ε near $\varepsilon = 0$.

Now we turn our attention to the eigenfunctions. We shall show that corresponding to an eigenvalue

$$\lambda(\varepsilon) = a_0 + a_1\varepsilon + \ldots ,$$

there exists an eigenvector $u(\varepsilon) = (u_1(\varepsilon),\ldots,u_n(\varepsilon))$, (i.e. a solution of the equation

$$\sum_{k=1}^{n} a_{ik}(\varepsilon)u_k(\varepsilon) = \lambda(\varepsilon)u_i(\varepsilon) \quad , \qquad i = 1,\ldots,n) \quad ,$$

each of whose components $u_k(\varepsilon)$, $k = 1,\ldots,n$, is a power series in ε convergent for small $|\varepsilon|$, and which is normalized,

$$\|u(\varepsilon)\| = \left(\sum_{k=1}^{n} |u_k(\varepsilon)|^2\right)^{\frac{1}{2}} = 1$$

for real ε for $|\varepsilon|$ small. Evidently this assertion is contained in the following statement:

Let $\gamma_{ik}(\varepsilon)$, $i,k = 1,\ldots,n$ be power series convergent in a neighborhood of $\varepsilon = 0$, and let

$$\det\,(\gamma_{ik}(\varepsilon)) = 0 \quad .$$

Then there exist power series $\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon)$, convergent in a neighborhood of $\varepsilon = 0$ such that

$$\sum_{k=1}^{n} \gamma_{ik}(\varepsilon)\alpha_k(\varepsilon) = 0 \quad , \qquad i = 1,\ldots,n \quad ,$$

and for real ε

$$\sum_{k=1}^{n} |\alpha_k(\varepsilon)|^2 = 1 \quad .$$

Remark: The previous assertion is obtained by setting $\gamma_{ik}(\varepsilon) = a_{ik}(\varepsilon) - \lambda(\varepsilon)\delta_{ik}(\varepsilon)$.

Proof: If all $\gamma_{ik}(\varepsilon)$ are identically zero, then $\alpha = (1,0,\ldots,0)$ satisfies all the requirements. Otherwise there exists an integer r, $1 \leq r \leq n-1$ such that some r by r minor determinant of $(\gamma_{ik}(\varepsilon))$ is not identically zero but every $r+1$ minor vanishes identically. There is no loss in generality in assuming that

$$\det\,(\gamma_{ik}(\varepsilon))_{i,k=1,\dots,r} \neq 0 \quad .$$

We denote by $\Gamma_{ki}(\varepsilon)$ the cofactor of $\gamma_{ik}(\varepsilon)$ in the determinant $\det\,(\gamma_{ik}(\varepsilon))_{i,k=1,\dots,r+1}$, and finally we define

$$f_k(\varepsilon) = \begin{cases} \Gamma_{k,r+1}(\varepsilon) & \text{for } k = 1,2,\dots,r+1 \\ 0 & \text{for } k = r+2,\dots,n \quad . \end{cases}$$

The $f_k(\varepsilon)$, $k = 1,\dots,n$, are convergent power series in ε for $|\varepsilon|$ small, and not all are identically zero because

$$f_{r+1}(\varepsilon) = \Gamma_{r+1,r+1}(\varepsilon) = \det\,(\gamma_{ik}(\varepsilon))_{i,k=1,\dots,r} \not\equiv 0 \quad .$$

Moreover we have

$$\sum_{k=1}^{n} \gamma_{ik}(\varepsilon) f_k(\varepsilon) = \sum_{k=1}^{r+1} \gamma_{ik}(\varepsilon)\Gamma_{k,r+1}(\varepsilon) = 0, \; i = 1,2,\dots,n \; ,$$

because the second term is equal to the determinant of the matrix obtained by replacing the $(r+1)$-st row of $(\gamma_{ik}(\varepsilon))_{i,k=1,\dots,r+1}$ by $\gamma_{i1}(\varepsilon),\gamma_{i2}(\varepsilon),\dots,\gamma_{ir+1}(\varepsilon)$. Now we have for real ε

$$|f_1|^2 + \dots + |f_n|^2 = \varepsilon^{2m}(b_0 + b_1\varepsilon + \dots) \quad , \quad b_0 \neq 0 \quad ,$$

where m is some integer and the power series converges for small $|\varepsilon|$; and consequently for real ε

$$\left(|f_1|^2 + \dots + |f_n|^2\right)^{\frac{1}{2}} = \varepsilon^m(d_0 + d_1\varepsilon + \dots), \quad d_0 \neq 0 \quad ,$$

where the power series converges for small $|\varepsilon|$. It is clear that the power series expansion of each $f_k(\varepsilon)$ begins at least with powers of order m. Define $\alpha_k = \alpha_k(\varepsilon)$ by

$$\alpha_k(\varepsilon) = \frac{f_k(\varepsilon)}{\varepsilon^m(d_0 + d_1\varepsilon + \dots)} \; , \quad k = 1,\dots,n \quad ;$$

evidently $\alpha_k(\varepsilon)$ are regular analytic functions ε for small $|\varepsilon|$, and $\sum_{k=1}^{n} |\alpha_k(\varepsilon)|^2 = 1$ for real ε. This completes the proof of our statement.

We are now in a position to formulate and prove

Theorem 1: *Suppose that $a_{ik}(\varepsilon)$, $i,k = 1,\dots,n$ are power series convergent in a neighborhood of $\varepsilon = 0$; and for real ε*

$$a_{ik}(\varepsilon) = \overline{a_{ki}(\varepsilon)} \quad .$$

Let $A(\varepsilon)$ denote the operator which transforms the vector $u = (u_1,\dots,u_n)$ into the vector $v = (v_1,\dots,v_n)$ according to the rule

$$v_i = \sum_{k=1}^{n} a_{ik}(\varepsilon)u_k \quad , \qquad i = 1,\dots,n \quad .$$

Suppose that λ is an eigenvalue of the operator $A = A(0)$ of multiplicity $h > 1$, and suppose the open interval $\lambda - d_1 < \mu < \lambda + d_2$, where d_1 and d_2 are positive numbers, contains no eigenvalue of A other than λ.

Then there exist power series

$$\lambda_1(\varepsilon),\dots,\lambda_h(\varepsilon) \quad ,$$

$$f_1^{(\nu)}(\varepsilon),\dots,f_n^{(\nu)}(\varepsilon) \quad , \qquad \nu = 1,\dots,h \quad ,$$

all convergent in a neighborhood of $\varepsilon = 0$, which satisfy the following conditions:

(1) *The vector $\phi^{(\nu)}(\varepsilon) = (f_1^{(\nu)}(\varepsilon),\dots,f_n^{(\nu)}(\varepsilon))$ is an eigenvector of $A(\varepsilon)$ belonging to the eigenvalue $\lambda_\nu(\varepsilon)$, i.e.*

$$A(\varepsilon)\phi^{(\nu)}(\varepsilon) = \lambda_\nu(\varepsilon)\phi^{(\nu)}(\varepsilon) \quad , \qquad \nu = 1,\dots,h \quad .$$

Furthermore $\lambda_\nu(0) = \lambda$, $\nu = 1,\ldots,h$; and for real ε the eigenvectors are orthonormal, i.e. for real ε

$$(\phi^{(\nu)}(\varepsilon), \phi^{(\mu)}(\varepsilon)) = \delta_{\nu\mu} \quad , \qquad \nu,\mu = 1,\ldots,h \quad .$$

(2) For each pair of positive numbers d_1', d_2' with $d_1' < d_1$, $d_2' < d_2$ there exists a positive number ρ such that the spectrum* of $A(\varepsilon)$ in the closed interval $\lambda - d_1' \leq \mu \leq \lambda + d_2'$ consists of the points $\lambda_1(\varepsilon),\ldots,\lambda_h(\varepsilon)$ provided $|\varepsilon| < \rho$.

Proof: We have already proved the first part of the theorem in case $h = 1$. Assume that part (1) of the conclusion is true for multiplicity $h-1$; we shall prove it for h.

By our preceding result we know the existence of an eigenvector $\phi^{(1)}(\varepsilon) = (f_1^{(1)}(\varepsilon),\ldots,f_n^{(1)}(\varepsilon))$ each of whose components is a power series in ε convergent for small $|\varepsilon|$ and such that $\|\phi^{(1)}(\varepsilon)\| = 1$ for real ε. Define the operator $P(\varepsilon)$ by

$$P(\varepsilon)u = (\phi^{(1)}(\varepsilon), u)\phi^{(1)}(\varepsilon) \quad ;$$

the matrix elements of $P(\varepsilon)$ are $p_{ik}(\varepsilon) = f_i^{(1)}(\varepsilon)\overline{f_k^{(1)}(\varepsilon)}$, $i,k = 1,\ldots,n$. $P(\varepsilon)$ is a projection operator which projects the n-dimensional space into the 1-dimensional "subspace" spanned by the vector $\phi^{(1)}(\varepsilon)$. Let us denote by $B(\varepsilon)$ the operator $B(\varepsilon) = A(\varepsilon) - P(\varepsilon)$, whose matrix elements $b_{ik}(\varepsilon) = a_{ik}(\varepsilon) - p_{ik}(\varepsilon)$ are power series in ε convergent for $|\varepsilon|$ small and $b_{ik}(\varepsilon) = \overline{b_{ki}(\varepsilon)}$ for ε real. Set $\psi_1 = \phi^{(1)}(0)$, and let $\psi_1, \psi_2, \ldots, \psi_h$ be an orthonormal set of eigenvectors belonging to the eigenvalue $\lambda = \lambda(0)$ of $A = A(0)$. Now we have

$$B(0)\psi_j = A(0)\psi_j - P(0)\psi_j = \lambda\psi_j - (\psi_1, \psi_j)\psi_1 = \lambda\psi_j \quad , \qquad j = 2,\ldots,h \quad .$$

---

* The spectrum of $A(\varepsilon)$ is the set of all eigenvalues of $A(\varepsilon)$.

Hence λ is an eigenvalue of $B(0)$ of multiplicity at least $h-1$. On the other hand we now show that the multiplicity does not exceed $h-1$. Otherwise there would exist an element ψ with $\|\psi\| = 1$ such that $B(0)\psi = \lambda\psi$ and $(\psi,\psi_j) = 0$, $j = 2,\ldots,h$. Since $B(0)\psi_1 = (\lambda-1)\psi_1$, it follows that $(\psi,\psi_1) = 0$, and hence $A(0)\psi = B(0)\psi + (\psi_1,\psi)\psi_1 = \lambda\psi$. Therefore $\psi,\psi_1,\psi_2,\ldots,\psi_h$ are $h+1$ linearly independent eigenvectors of $A(0)$ corresponding to the eigenvalue λ. But this is impossible because the multiplicity of λ is h, and therefore the multiplicity of the eigenvalue λ of $B(0)$ is exactly $h-1$.

By our assumption, there exist power series

$$\lambda_\nu(\varepsilon), f_1^{(\nu)}(\varepsilon), \ldots, f_n^{(\nu)}(\varepsilon) \quad , \qquad \nu = 2,\ldots,h \quad ,$$

convergent in a neighborhood of $\varepsilon = 0$ which satisfy the relations

$$B(\varepsilon)\phi^{(\nu)}(\varepsilon) = \lambda_\nu(\varepsilon)\phi^{(\nu)}(\varepsilon) \quad , \qquad \nu = 2,\ldots,h \quad ,$$

where $\phi^{(\nu)}(\varepsilon) = (f_1^{(\nu)}(\varepsilon),\ldots,f_n^{(\nu)}(\varepsilon))$, and for real ε

$$(\phi^{(\nu)}(\varepsilon),\phi^{(\mu)}(\varepsilon)) = \delta_{\nu\mu} \quad , \qquad \nu,\mu = 2,\ldots,h \quad .$$

Moreover we find

$$A(\varepsilon)\phi^{(1)}(\varepsilon) = \lambda_1(\varepsilon)\phi^{(1)}(\varepsilon), \text{ and } \quad B(\varepsilon)\phi^{(1)}(\varepsilon) = (\lambda_1(\varepsilon)-1)\phi^{(1)}(\varepsilon) \quad .$$

For $|\varepsilon|$ small we certainly have $\lambda_\nu(\varepsilon) \neq \lambda_1(\varepsilon)-1$, $\nu = 2,\ldots,h$, and consequently $(\phi^{(1)}(\varepsilon),\phi^{(\nu)}(\varepsilon)) = 0$, $\nu = 2,\ldots,h$. Hence we finally obtain for $\nu = 2,\ldots,h$,

$$\begin{aligned} A(\varepsilon)\phi^{(\nu)}(\varepsilon) &= B(\varepsilon)\phi^{(\nu)}(\varepsilon) + P(\varepsilon)\phi^{(\nu)}(\varepsilon) \\ &= \lambda_\nu(\varepsilon)\phi^{(\nu)}(\varepsilon) + (\phi^{(1)}(\varepsilon),\phi^{(\nu)}(\varepsilon))\phi^{(1)}(\varepsilon) \\ &= \lambda_\nu(\varepsilon)\phi^{(\nu)}(\varepsilon) \quad . \end{aligned}$$

Therefore the first part of the theorem is proved.

In exactly the same way we can treat each eigenvalue of $A(0)$. Finally we obtain n convergent power series which can be labeled $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$ and which for fixed ε is the whole spectrum of $A(\varepsilon)$. Part (2) of the theorem is then an immediate consequence. Theorem 1 is completely proved.

It is interesting to observe what we have not proved. We do not claim that if λ is an unperturbed eigenvalue of multiplicity h and if $\psi_1,\ldots,\psi_h$ are an orthonormal set of eigenvectors of the unperturbed operator belonging to λ then there exist eigenvectors $\phi^{(\nu)}(\varepsilon)$, $\nu = 1,\ldots,h$, which are convergent power series in ε and such that the vector equation

$$\phi^{(\nu)}(\varepsilon) = \psi_\nu + \psi_{1,\nu}\varepsilon + \psi_{2,\nu}\varepsilon^2 + \ldots$$

obtains. All that has been proved is that the $\phi^{(\nu)}(\varepsilon)$ exist and $\phi^{(\nu)}(0)$, $\nu = 1,\ldots,h$, are an orthonormal set of eigenvectors of the perturbed operators. In general $\phi^{(\nu)}(0)$ cannot be prescribed in advance; the perturbation method itself selects them. Indeed this is all that can be expected.

Consider for example the Hermitian matrix

$$\begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$$

in the (complex) 2-dimensional space. For $\varepsilon = 0$ every non-zero vector is an eigenvector; in particular we can take

$$\psi_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) , \quad \psi_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) .$$

For $\varepsilon \neq 0$ the normalized eigenvectors are uniquely determined up to a factor of unit magnitude as

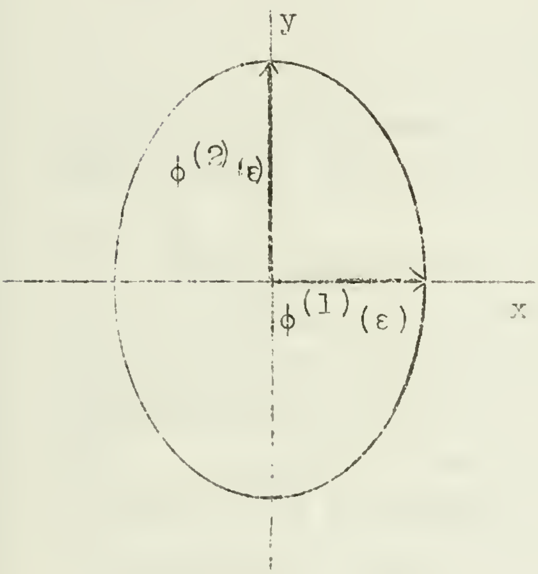
$$\phi^{(1)}(\varepsilon) = (0,1) , \quad \phi^{(2)}(\varepsilon) = (1,0) ,$$

which are quite unrelated to ψ_1, ψ_2.

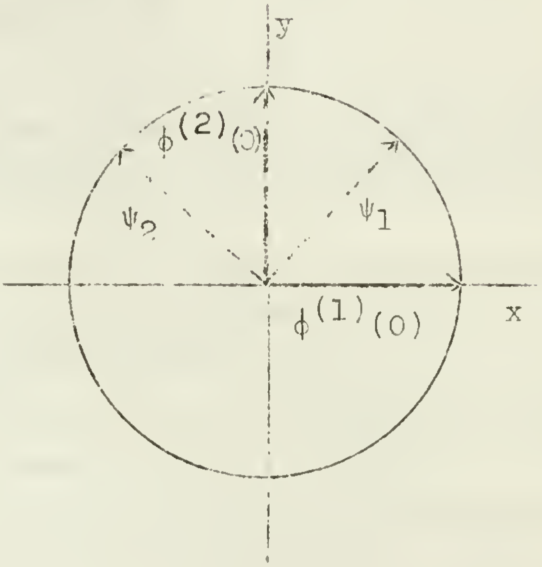
We can give a more geometrical description of this example if we restrict ourselves to the real 2-dimensional space, say the x,y-plane. The eigenvalue problem is the same as finding the principal axes of the ellipse

$$(1+\varepsilon)x^2 + (1-\varepsilon)y^2 = 1 \quad ;$$

we take ε real and $-1 < \varepsilon < 1$.



$0 < \varepsilon < 1$



$\varepsilon = 0$

For $\varepsilon = 0$ the ellipse goes into the unit circle. As principal axes we could have taken ψ_1, ψ_2 as well as $\phi^{(1)}(0)$, $\phi^{(2)}(0)$; but the latter are essentially the only ones to which the perturbed eigenvalues can be joined analytically.

§2. Several analytic parameters.

The matrix

$$\begin{pmatrix} 1+2\varepsilon_1 & \varepsilon_1+\varepsilon_2 \\ \varepsilon_1+\varepsilon_2 & 1+2\varepsilon_2 \end{pmatrix}$$

is a Hermitian matrix for real ε_1, ε_2 whose elements $a_{ik}(\varepsilon_1,\varepsilon_2)$ are convergent power series of ε_1, ε_2 for all values of ε_1, ε_2. The eigenvalues of this matrix are

$$\lambda = 1+\varepsilon_1+\varepsilon_2 \pm \sqrt{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} \quad .$$

Obviously no arrangement of these eigenvalues could make them power series in ε_1 and ε_2 convergent in some neighborhood of $\varepsilon_1 = \varepsilon_2 = 0$.

This example shows that it is not true in general that the eigenvalues of a matrix $(a_{ik}(\varepsilon_1,\varepsilon_2))$, whose elements $a_{ik}(\varepsilon_1,\varepsilon_2)$ are convergent power series of ε_1, ε_2 in a neighborhood of $\varepsilon_1 = \varepsilon_2 = 0$ with $a_{ik}(\varepsilon_1,\varepsilon_2) = \overline{a_{ki}(\varepsilon_1,\varepsilon_2)}$ for real ε_1, ε_2, can be considered power series in ε_1, ε_2 convergent in a neighborhood of $\varepsilon_1 = \varepsilon_2 = 0$. In general the eigenvalues are the roots of the secular equation

$$(-1)^n \det(a_{ik}(\varepsilon_1,\varepsilon_2)-\lambda\delta_{ik}) = \lambda^n+c_1(\varepsilon_1,\varepsilon_2)\lambda^{n-1}+\dots+c_n(\varepsilon_1,\varepsilon_2) = 0 \quad ,$$

where the coefficients are power series in ε_1, ε_2 convergent in a neighborhood of $\varepsilon_1 = \varepsilon_2 = 0$. The possible singularities of a root $\lambda = \lambda(\varepsilon_1,\varepsilon_2)$ are consequently those of an algebraic function of two variables. The algebraic equation is special in that it is the characteristic equation of a Hermitian matrix. Therefore all the roots are real when ε_1 and ε_2 are real. In the case of one parameter ε this is the only restriction needed to prove that a polynomial equation in λ, say

$$\lambda^n + c_1(\varepsilon)\lambda^{n-1} + \dots + c_n(\varepsilon) = 0 \quad ,$$

with coefficients convergent power series in ε in a neighborhood of $\varepsilon = 0$, is the secular equation of a matrix whose elements are convergent power series of ε for small $|\varepsilon|$ and which is Hermitian for ε real. Indeed we have already proved that the roots $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$ are convergent power series in ε. Thus

$$\lambda^n+c_1(\varepsilon)\lambda^{n-1}+\ldots+c_n(\varepsilon) = \prod_{i=1}^{n} (\lambda-\lambda_i(\varepsilon)) = (-1)^n \det((\lambda_i(\varepsilon)-\lambda)\delta_{ik}) \quad ,$$

i.e. our equation is the secular equation of the diagonal matrix whose diagonal elements are $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$.

We now consider the same question for two parameters ε_1, ε_2. Can one write a polynomial

$$\lambda^n + c_1(\varepsilon_1,\varepsilon_2)\lambda^{n-1} + \ldots + c_n(\varepsilon_1,\varepsilon_2) \quad ,$$

whose coefficients are power series in ε_1, ε_2 convergent in a neighborhood of $\varepsilon_1 = \varepsilon_2 = 0$ and whose zeros are all real when ε_1 and ε_2 are real, in the form

$$\lambda^n+c_1(\varepsilon_1,\varepsilon_2)\lambda^{n-1}+\ldots+c_n(\varepsilon_1,\varepsilon_2) \equiv (-1) \det (a_{ik}(\varepsilon_1,\varepsilon_2)-\lambda\delta_{ik})$$

where the $a_{ik}(\varepsilon_1,\varepsilon_2)$ are power series in ε_1, ε_2 convergent with $a_{ik}(\varepsilon_1,\varepsilon_2) = \overline{a_{ki}(\varepsilon_1,\varepsilon_2)}$ for real ε_1, ε_2? The answer to this question in general does not seem to be known.

In case $n = 2$, the answer is yes. Suppose that

$$\lambda^2 + p(\varepsilon_1,\varepsilon_2)\lambda + q(\varepsilon_1,\varepsilon_2)$$

is a polynomial in λ of degree 2 whose zeros are real for real ε_1, ε_2 and where p, q are convergent power series in ε_1, ε_2 for small $|\varepsilon_1|$, $|\varepsilon_2|$. We must prove the existence of power series $a(\varepsilon_1,\varepsilon_2)$, $b(\varepsilon_1,\varepsilon_2)$, $\alpha(\varepsilon_1,\varepsilon_2)$, $\beta(\varepsilon_1,\varepsilon_2)$ with real coefficients and convergent for small $|\varepsilon_1|$, $|\varepsilon_2|$ such that

$$\lambda^2 + p\lambda + q \equiv \begin{vmatrix} a - \lambda & \alpha + i\beta \\ \alpha - i\beta & b - \lambda \end{vmatrix} .$$

If such a, b, α, β exist, then they satisfy the relations

$$a + b = -p \quad , \quad ab - \alpha^2 - \beta^2 = q \quad ,$$

or

$$(a - b)^2 + 4\alpha^2 + 4\beta^2 = p^2 - 4q \quad .$$

Now $p^2 - 4q = F(\varepsilon_1, \varepsilon_2)$ is a power series in ε_1, ε_2 convergent for small $|\varepsilon_1|$, $|\varepsilon_2|$ which is real and non-negative for real ε_1, ε_2. Anticipating the lemma stated at the end of this section, we may conclude that there exist two power series $G(\varepsilon_1, \varepsilon_2)$ and $H(\varepsilon_1, \varepsilon_2)$ with real coefficients and convergent for small $|\varepsilon_1|$, $|\varepsilon_2|$ and such that

$$F(\varepsilon_1, \varepsilon_2) = [G(\varepsilon_1, \varepsilon_2)]^2 + [H(\varepsilon_1, \varepsilon_2)]^2 \quad .$$

We then set

$$\beta = 0 \ , \quad \alpha = \frac{H}{2} \ , \quad a = \frac{G - p}{2} \ , \quad b = -\frac{G + p}{2} \quad ,$$

which provides a solution.

For s parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s$ with $s > 2$, it is not always possible to write $F(\varepsilon_1, \ldots, \varepsilon_s)$, which is supposed to be a convergent power series with real coefficients and which is non-negative for real $\varepsilon_1, \ldots, \varepsilon_s$, as a sum of three squares of convergent power series with real coefficients. Consequently it is not possible in general to find a, b, α, β such that

$$F(\varepsilon_1, \ldots, \varepsilon_s) = p^2 - 4q = (a - b)^2 + 4\alpha^2 + 4\beta^2 \quad .$$

Thus the fact that the eigenvalues $\lambda(\varepsilon_1, \ldots, \varepsilon_s)$ of a Hermitian matrix are roots of an equation

$$\det\,(a_{ik} - \lambda\delta_{ik}) = 0$$

is (for $s > 2$) a more stringent restriction than that which arises from the real character of the roots for $\varepsilon_1, \ldots, \varepsilon_s$ all real, and which excluded singularities altogether in the case of one parameter. It would be very interesting to know what singularities are possible when there are an arbitrary number of variables.

The simple remarks made above indicate that perturbation theory for more than one perturbation parameter is a field open for research.

We conclude this section with the lemma used above.

Lemma:[*] *Let $F(\varepsilon_1, \varepsilon_2)$ be a power series with real coefficients convergent for small $|\varepsilon_1|$, $|\varepsilon_2|$ and, suppose $F(\varepsilon_1, \varepsilon_2)$ is non-negative for real ε_1, ε_2. Then there exist two power series $G(\varepsilon_1, \varepsilon_2)$ and $H(\varepsilon_1, \varepsilon_2)$ with real coefficients, converging for small $|\varepsilon_1|$, $|\varepsilon_2|$ such that*

$$F = G^2 + H^2 \quad .$$

§3. *Non-analytic perturbation.*

Let $A(\varepsilon)$ be an operator which is represented by the Hermitian matrix $(a_{ik}(\varepsilon))$ whose elements are supposed to be complex-valued continuous functions of the real variable ε for ε in the open interval $\varepsilon' < \varepsilon < \varepsilon''$. It is well known that the eigenvalues of such a matrix are continuous functions of ε in $\varepsilon' < \varepsilon < \varepsilon''$ when ordered according to magnitude, say,

$$\lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \ldots \leq \lambda_n(\varepsilon) \quad .$$

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[*] F. Rellich, *Aufgabe 267*, Jahresbericht der Deutchen Mathematiker Vereinigung, vol. 48 (1938). A proof was given by H. Kneser, *Lösung der Aufgabe 267*, ibid., vol. 50 (1940), pp. 29-31. We shall give the proof in an appendix to these notes.

The proof runs as follows:

Denote by $\lambda = \lambda(\varepsilon)$ any one of the eigenvalues $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$. Then for each ε in $(\varepsilon',\varepsilon'')$ there is an eigenelement $u = (u_1,\ldots,u_n)$ such that

$$\lambda u_i = \sum_{k=1}^{n} a_{ik} u_k \quad , \qquad i = 1,\ldots,n \quad ,$$

and

$$\sum_{k=1}^{n} |u_i|^2 = 1 \quad .$$

Hence we have

$$\lambda = \sum_{i,k=1}^{n} a_{ik} u_k \bar{u}_i \quad .$$

Suppose ε_1 and ε_2 are real numbers such that $\varepsilon' < \varepsilon_1 < \varepsilon_2 < \varepsilon''$. Since the $a_{ik}(\varepsilon)$ are continuous in $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$, there is a number M such that

$$|a_{ik}(\varepsilon)| \leq M \quad , \qquad i,k = 1,\ldots,n \quad ,$$

in $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$. Consequently we obtain the estimate: $|\lambda| \leq nM$ for $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$, i.e. all the eigenvalues $\lambda_1(\varepsilon),\ldots,\lambda_n(\varepsilon)$ are uniformly bounded in $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$. Now we wish to prove for any number ε_o with $\varepsilon' < \varepsilon_o < \varepsilon''$ that

$$\lim_{\nu \to \infty} \lambda_j(\varepsilon_o + h_\nu) = \lambda_j(\varepsilon_o) \quad , \qquad j = 1,\ldots,n \quad ,$$

if h_ν is any sequence of real numbers tending to 0 as $\nu \to \infty$. Since $\varepsilon_o + h_\nu$, except for a finite number of ν, lies in a fixed closed bounded subinterval of $\varepsilon' < \varepsilon < \varepsilon''$, the sequences

$$\lambda_j^{(\nu)} = \lambda_j(\varepsilon_o + h_\nu)$$

are uniformly bounded. Therefore one can select a subsequence of the integers, say ν_m, such that $\lambda_j^{(\nu_m)}$ are all convergent. Let us put

$$\lim_{m \to \infty} \lambda_j^{(\nu_m)} = \mu_j \quad .$$

We thus obtain

$$(-1)^n \det \left(a_{ik}(\varepsilon_o + h_{\nu_m}) - \lambda\delta_{ik}\right) = \left(\lambda-\lambda_1^{(\nu_m)}\right)\cdots\left(\lambda-\lambda_n^{(\nu_m)}\right) ,$$

and letting $m \longrightarrow \infty$, we find

$$(-1)^n \det \left(a_{ik}(\varepsilon_o) - \lambda\delta_{ik}\right) = (\lambda-\mu_1)\cdots(\lambda-\mu_n) .$$

Therefore the set of numbers $\lambda_1(\varepsilon_o),\ldots,\lambda_n(\varepsilon_o)$ must coincide with $\mu_1,\ldots,\mu_n$. Since $\lambda_1^{(\nu_m)} \leq \cdots \leq \lambda_n^{(\nu_m)}$, it is clear that $\mu_1 \leq \cdots \leq \mu_n$; and hence

$$\lambda_j(\varepsilon_o) = \mu_j \quad , \qquad j = 1,\ldots,n \quad ,$$

or

$$\lambda_j(\varepsilon_o) = \lim_{m\longrightarrow\infty} \lambda_j(\varepsilon_o + h_{\nu_m}) \quad , \quad j = 1,\ldots,n \quad .$$

It therefore follows that $\lim_{\nu\longrightarrow\infty} \lambda_j(\varepsilon_o + h_\nu) = \lambda_j(\varepsilon_o)$, $j = 1,\ldots,n$.

Although the eigenvalues (suitably ordered) are continuous functions of the perturbation parameter when the perturbation is continuous, it is impossible in general to find a complete orthonormal set of eigenvectors whose components are continuous functions of the perturbation parameter. Consider the following example: for $\varepsilon \neq 0$,

$$(a_{ik}(\varepsilon)) = \begin{pmatrix} 1 - e^{-\frac{1}{\varepsilon^2}} \cos \frac{2}{\varepsilon} & - e^{-\frac{1}{\varepsilon^2}} \sin \frac{2}{\varepsilon} \\ - e^{-\frac{1}{\varepsilon^2}} \sin \frac{2}{\varepsilon} & 1 - e^{-\frac{1}{\varepsilon^2}} \cos \frac{2}{\varepsilon} \end{pmatrix} ,$$

and

$$(a_{ik}(0)) = (\delta_{ik}) \quad .$$

The elements $a_{ik}(\varepsilon)$ are continuous functions of ε for $-\infty < \varepsilon < \infty$. The corresponding operator $A(\varepsilon)$ on the two dimensional space has eigenvectors for $\varepsilon \neq 0$

$$\phi^{(1)}(\varepsilon) = (\cos \frac{1}{\varepsilon},\ \sin \frac{1}{\varepsilon}) \quad ,$$

$$\phi^{(2)}(\varepsilon) = (\sin \frac{1}{\varepsilon}, - \cos \frac{1}{\varepsilon}) \quad ,$$

belonging to the eigenvalues

$$\lambda_1(\varepsilon) = 1 - e^{-\frac{1}{\varepsilon^2}} \quad ,$$

$$\lambda_2(\varepsilon) = 1 + e^{-\frac{1}{\varepsilon^2}} \quad ,$$

respectively. Since $\lambda_1(\varepsilon) \neq \lambda_2(\varepsilon)$ for $\varepsilon \neq 0$, if $u^{(j)}(\varepsilon)$, $j = 1,2$, are any normalized eigenvectors belonging to $\lambda_j(\varepsilon)$, $j = 1,2$, respectively, then $u^{(j)} = e^{i\alpha_j(\varepsilon)} \phi^{(j)}(\varepsilon)$, $j = 1,2$, where $\alpha_j(\varepsilon)$, $j = 1,2$, are real for real $\varepsilon \neq 0$. It is obviously impossible to define $\alpha_1(\varepsilon)$, say, in such a way that $u^{(1)}(\varepsilon)$ tends to a limit as $\varepsilon \to 0$, because

$$\left| e^{i\alpha_1(\varepsilon)} \cos \frac{1}{\varepsilon} \right| = \left| \cos \frac{1}{\varepsilon} \right|$$

has no limit as $\varepsilon \to 0$.

Note in this example that $a_{ik}(\varepsilon)$ is not only continuous but also has continuous derivatives of all orders.

Since in general the individual eigenvectors do not depend continuously on the perturbation parameter ε even though the operator $A(\varepsilon)$ does, it is necessary to work, not with an eigenvector, but rather with the space spanned by all the eigenvectors belonging to the same eigenvalue.

Consider a Hermitian operator A on the n-dimensional space $\mathfrak{H}^n$. We shall denote by $\mathfrak{H}^n_\lambda$ the subspace of all eigenvectors of A belonging to eigenvalues which do not exceed λ. Suppose that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of A and $\phi^{(1)},\ldots,\phi^{(n)}$ is an orthonormal set of corresponding eigenvectors. The subspace $\mathfrak{H}^n_\lambda$ for $\lambda < \lambda_1$ consists of the null vector, and for $\lambda \geq \lambda_n$ is the whole space $\mathfrak{H}^n$. We define the operator E_λ by

$$E_\lambda u = \sum_{\lambda_j \leq \lambda} (\phi^{(j)},u)\phi^{(j)} \quad ;$$

E_λ projects $\mathfrak{H}^n$ into $\mathfrak{H}^n_\lambda$. Evidently $E_\lambda = 0$ if $\lambda < \lambda_1$ and $E_\lambda = 1$, the identity, if $\lambda \geq \lambda_n$. If the interval $\lambda' \leq \lambda \leq \lambda''$ does not contain any of the eigenvalues, then $E_{\lambda'} = \lambda_{\lambda''}$. Notice that whereas the $\phi^{(1)},\ldots,\phi^{(n)}$ are not uniquely determined, the projector E_λ is unique.

If $A = A(\varepsilon)$ depends on a parameter ε, then so does E_λ; and one might ask, for example, whether or not E_λ depends continuously on ε if A does. (Continuity of $E_\lambda(\varepsilon)$ means continuity of the elements of the matrix representing $E_\lambda(\varepsilon)$.) Evidently $E_\lambda(\varepsilon)$ will not depend continuously on ε at $\varepsilon = \varepsilon_o$ if λ is one of the eigenvalues $\lambda_1(\varepsilon_o) \leq \ldots \leq \lambda_n(\varepsilon_o)$ of $A(\varepsilon_o)$. If $\lambda \neq \lambda_j(\varepsilon_o)$, $j = 1,\ldots,n$, however, we can establish the continuous dependence of $E_\lambda(\varepsilon)$ on ε at $\varepsilon = \varepsilon_o$. We shall defer the proof to a more general case in Chapter II.

§4. The least eigenvalue is a convex function of a linear perturbation parameter.

Suppose that (a_{ik}) and (b_{ik}), $i,k = 1,\ldots,n$ are two Hermitian matrices, and let $\lambda(\varepsilon)$, $-\infty < \varepsilon < +\infty$, be the least eigenvalue of the Hermitian matrix $(a_{ik} + \varepsilon b_{ik})$. Then $\lambda(\varepsilon)$ is a convex function of ε.

Proof:[*] Suppose that we denote by A and B the Hermitian operators corresponding to (a_{ik}) and (b_{ik}) respectively. For any real ε we denote by ϕ_o a normalized eigenvector of $A + \varepsilon_o B$ belonging to $\lambda(\varepsilon_o)$. Thus we have

$$A\phi_o + \varepsilon_o B\phi_o = \lambda(\varepsilon_o)\phi_o$$

and

$$a + \varepsilon_o b = \lambda(\varepsilon_o) \quad ,$$

where $a = (\phi_o, A\phi_o)$ and $b = (\phi_o, B\phi_o)$. Since

$$(\phi, A\phi) + \varepsilon(\phi, B\phi) \geq \lambda(\varepsilon)$$

for all vectors ϕ of unit length, it follows that

$$a + \varepsilon b \geq \lambda(\varepsilon) \quad ,$$

for all real ε. This inequality together with the fact that the equality holds at $\varepsilon = \varepsilon_o$ shows that the line $y = a + \varepsilon b$ is a line of support of the curve $y = \lambda(\varepsilon)$ at $\varepsilon = \varepsilon_o$. Since ε_o was arbitrary, the curve $y = \lambda(\varepsilon)$ has a line of support at each point; and therefore $\lambda(\varepsilon)$ is convex.

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[*] The lecturer learned this proof from G. Polya. See P. R. Garabedian and M. Schiffer, "Convexity of Domain Functionals", Journal d'Analyse Mathematique, Jerusalem, 1952/53.

§5. Differentiable dependence on the perturbation parameter.

In an example of §3 we exhibited a 2 × 2 Hermitian matrix, whose elements have continuous derivatives of all orders with respect to the (real) perturbation parameter ε, but whose eigenvectors are not even continuous functions of ε. However the eigenvalues have continuous derivatives of all orders with respect to the real variable ε. This result suggests that the eigenvalues reflect the regularity of the matrix elements whereas individual eigenvectors may behave quite badly.

We suppose that the functions $a_{ik}(\varepsilon)$, $i,k = 1,\dots n$, which are such that $a_{ik}(\varepsilon) = \overline{a_{ki}(\varepsilon)}$ for $\varepsilon' < \varepsilon < \varepsilon''$, have continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$. In the following we shall talk of an Hermitian operator $A(\varepsilon)$ on $\mathfrak{H}^n$ for which $\lim\limits_{t \to 0} \frac{A(\varepsilon + t) - A(t)}{t} = A'(\varepsilon)$ exists and is a continuous function on ε. Our question is whether or not there exists an arrangement $\lambda_1(\varepsilon),\dots,\lambda_n(\varepsilon)$ of the eigenvalues of the Hermitian matrix $(a_{ik}(\varepsilon))$ such that the n functions $\lambda_1(\varepsilon),\dots,\lambda_n(\varepsilon)$ have continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$.

Let us examine the case n = 2. In this case the eigenvalues are the solutions of the quadratic equation

$$\begin{vmatrix} a(\varepsilon) - \lambda & \alpha(\varepsilon) + i\beta(\varepsilon) \\ \alpha(\varepsilon) - i\beta(\varepsilon) & b(\varepsilon) - \lambda \end{vmatrix} \equiv \lambda^2 - (a+b)\lambda + ab - \alpha^2 - \beta^2 = 0 \quad ,$$

where $a(\varepsilon)$, $b(\varepsilon)$, $\alpha(\varepsilon)$, $\beta(\varepsilon)$ are real functions with continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$. For each ε the numbers

$$\tfrac{1}{2}(a+b) \pm \tfrac{1}{2}[(a-b)^2 + \alpha^2 + \beta^2]^{\frac{1}{2}}$$

are the eigenvalues. Are these functions $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ with continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$ which coincide with these two numbers, apart from order? The

answer is *yes*, and the proof follows from the following simple statement (which is contained in a more general statement made on page 63).

If $f(\varepsilon)$ and $g(\varepsilon)$ are real functions with continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$, then there exists a real function $c(\varepsilon)$ with a continuous first derivative with

$$(c(\varepsilon))^2 = (f(\varepsilon))^2 + (g(\varepsilon))^2$$

for $\varepsilon' < \varepsilon < \varepsilon''$.

Of course we can immediately extend this statement to any finite number of functions. If $f_1(\varepsilon), f_2(\varepsilon), \ldots, f_s(\varepsilon)$ are real functions with continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$ then there exists a real function $c(\varepsilon)$ with a continuous first derivative with

$$(c(\varepsilon))^2 = (f_1(\varepsilon))^2 + (f_2(\varepsilon))^2 + \ldots + (f_s(\varepsilon))^2$$

for $\varepsilon' < \varepsilon < \varepsilon''$. In particular there exists a real function $c(\varepsilon)$ with a continuous first derivative in $\varepsilon' < \varepsilon < \varepsilon''$ such that

$$(c(\varepsilon))^2 = (a-b)^2 + \alpha^2 + \beta^2 \quad .$$

If we set

$$\lambda_1(\varepsilon) = \frac{a+b+c}{2} , \quad \lambda_2(\varepsilon) = \frac{a+b-c}{2} ,$$

then $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ have continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$ and for each ε their values are all the eigenvalues of the matrix.

Now we formulate the main

Theorem: *Suppose that $A(\varepsilon)$ is a family of Hermitian operators defined on the n-dimensional Hilbert space $\mathfrak{H}^n$ and continuous in $\varepsilon' < \varepsilon < \varepsilon''$. Suppose that*

$$\lim_{t \to 0} \frac{A(\varepsilon + t) - A(\varepsilon)}{t} = A'(\varepsilon)$$

exists for each ε in $\varepsilon' < \varepsilon < \varepsilon''$, and $A'(\varepsilon)$ is Hermitian and continuous in $\varepsilon' < \varepsilon < \varepsilon''$. Then there exist functions $\lambda_1(\varepsilon),\dots,\lambda_n(\varepsilon)$ with continuous first derivatives in $\varepsilon' < \varepsilon < \varepsilon''$ such that

$$A(\varepsilon)\phi^{(j)}(\varepsilon) = \lambda_j(\varepsilon)\phi^{(j)}(\varepsilon) \quad , \qquad j = 1,\dots,n \quad ,$$

for some (properly chosen) orthonormal system of vector-functions $\phi^{(1)}(\varepsilon),\dots,\phi^{(n)}(\varepsilon)$.

The proof of this theorem is divided into two parts. The first part (Theorem 1, below) uses the fact that the eigenvalues are not only the solutions of an algebraic equation of degree n, but more particularly the solutions of the secular equation of a Hermitian matrix. The second part (Auxiliary Theorem, below) is not directly concerned with eigenvalue problems, but is a general statement about functions of a real variable.

One can relax the requirement that the operators be Hermitian, but some condition must be imposed for the conclusion to hold true. Consider the example

$$(a_{ik}(\varepsilon)) = \begin{pmatrix} 0 & |\varepsilon|^{\frac{3}{2}} \\ 1 & 0 \end{pmatrix} \quad ,$$

a 2 X 2 matrix whose elements have continuous first derivatives in $-\infty < \varepsilon < \infty$. The eigenvalues of this matrix are

$$\mu_1(\varepsilon) = -|\varepsilon|^{\frac{3}{4}} \quad , \qquad \mu_2(\varepsilon) = |\varepsilon|^{\frac{3}{4}} \quad ;$$

and it is not possible to "arrange" the eigenvalues to be functions with continuous first derivatives in $-\infty < \varepsilon < \infty$.

We shall first prove

Theorem 1: Suppose that $A(\varepsilon)$ is a family of Hermitian operators defined on the n-dimensional Hilbert space $\mathfrak{H}^n$ and continuous for $\varepsilon' < \varepsilon < \varepsilon''$. Suppose that

$$\lim_{t \to 0} \frac{A(\varepsilon + t) - A(\varepsilon)}{t} = A'(\varepsilon)$$

exists for each ε in $\varepsilon' < \varepsilon < \varepsilon''$, and $A'(\varepsilon)$ is Hermitian and continuous in $\varepsilon' < \varepsilon < \varepsilon''$. Let $\phi^{(1)}(\varepsilon), \ldots, \phi^{(n)}(\varepsilon)$ be an orthonormal set of eigenvectors of $A(\varepsilon)$ such that

$$A(\varepsilon)\phi^{(j)}(\varepsilon) = \mu_j(\varepsilon)\phi^{(j)}(\varepsilon) \quad , \quad j = 1, \ldots, n \quad ,$$

and

$$\mu_1(\varepsilon) \leq \mu_2(\varepsilon) \leq \ldots \leq \mu_n(\varepsilon) \quad .$$

For a fixed ε in the interval $\varepsilon' < \varepsilon < \varepsilon''$, consider any arbitrary eigenvalue $\mu_p(\varepsilon)$. Suppose that $\mu_p(\varepsilon)$ has multiplicity h; say $\mu_j(\varepsilon) = \mu_p(\varepsilon)$ for $p \leq j \leq p+h-1$, and $\mu_j(\varepsilon) \neq \mu_p(\varepsilon)$ for $1 \leq j \leq p-1$ and $p+h \leq j \leq n$. Let us denote by $\mathfrak{M}$ the h-dimensional subspace spanned by $\phi^{(j)}(\varepsilon)$, $j = p, \ldots, p+h-1$, and let us denote by $P(\varepsilon)$ the projector of $\mathfrak{H}^n$ into $\mathfrak{M}$.

Then we can conclude:

(1) If τ_m is a sequence of non-zero real numbers tending to 0 for which $\lim_{m \to 0} \phi^{(j)}(\varepsilon + \tau_m) = \psi^{(j)}(\varepsilon)$, $j = p, \ldots, p+h-1$, exists, then

$$\lim_{m \to \infty} \frac{\mu_j(\varepsilon + \tau_m) - \mu_j(\varepsilon)}{\tau_m} = \rho_j(\varepsilon) \quad , \quad j = p, \ldots, p+h-1$$

also exist; and

$$A(\varepsilon)\psi^{(j)}(\varepsilon) = \mu_j(\varepsilon)\psi^{(j)}(\varepsilon) \quad , \qquad j = p,\ldots,p+h-1 \quad ,$$

$$P(\varepsilon)A'(\varepsilon)\psi^{(j)}(\varepsilon) = \rho_j(\varepsilon)\psi^{(j)}(\varepsilon) \quad , \qquad j = p,\ldots,p+h-1 \quad ,$$

$$(\psi^{(k)}(\varepsilon),\psi^{(j)}(\varepsilon)) = \delta_{kj} \quad , \qquad k,j = p,\ldots,p+h-1 \quad .$$

(2) $$\lim_{\substack{t \to 0 \\ t > 0}} \frac{\mu_j(\varepsilon + t) - \mu_j(\varepsilon)}{t} = r_j(\varepsilon), \quad \lim_{\substack{t \to 0 \\ t < 0}} \frac{\mu_j(\varepsilon + t) - \mu_j(\varepsilon)}{t} = s_j(\varepsilon)$$

exist, for $j = p,\ldots,p+h-1$. The three sets of numbers $\rho_p(\varepsilon),\ldots,\rho_{p+h-1}(\varepsilon)$ and $r_p(\varepsilon),\ldots,r_{p+h-1}(\varepsilon)$ and $s_p(\varepsilon),\ldots,s_{p+h-1}(\varepsilon)$ are the same, apart from order.

(3) *If $\lim_{\nu \to \infty} \varepsilon_\nu = \varepsilon$ and $\varepsilon' < \varepsilon_\nu < \varepsilon''$, then the set of numbers $r_p(\varepsilon_\nu),\ldots,r_{p+h-1}(\varepsilon_\nu)$ have at most n cluster points, namely, $r_p(\varepsilon),\ldots,r_{p+h-1}(\varepsilon)$.*

Proof: Since $(\phi^{(k)}(\varepsilon + \tau_m),\phi^{(j)}(\varepsilon + \tau_m)) = \delta_{kj}$, it follows that $(\psi^{(k)}(\varepsilon),\psi^{(j)}(\varepsilon)) = \delta_{kj}$, $k,j = p,\ldots,p+h-1$; and since $A(\varepsilon + \tau_m)\phi^{(j)}(\varepsilon + \tau_m) = \mu_j(\varepsilon + \tau_m)\phi^{(j)}(\varepsilon + \tau_m)$, it follows that $A(\varepsilon)\psi^{(j)}(\varepsilon) = \mu_j(\varepsilon)\psi^{(j)}(\varepsilon)$, $j = p,\ldots,p+h-1$. Moreover from

$$(A(\varepsilon) - \mu_j(\varepsilon))\phi^{(j)}(\varepsilon + \tau_m) + (A(\varepsilon + \tau_m) - A(\varepsilon))\phi^{(j)}(\varepsilon + \tau_m)$$
$$- (\mu_j(\varepsilon + \tau_m) - \mu_j(\varepsilon))\phi^{(j)}(\varepsilon + \tau_m) = 0 \quad ,$$

it follows that

$$(\psi, \frac{A(\varepsilon + \tau_m) - A(\varepsilon)}{\tau_m} \phi^{(j)}(\varepsilon + \tau_m))$$

$$- \frac{\mu_j(\varepsilon + \tau_m) - \mu_j(\varepsilon)}{\tau_m}(\psi, \phi^{(j)}(\varepsilon + \tau_m)) = 0$$

for any element ψ in $\mathfrak{M}$. For $\psi = \psi^{(j)}(\varepsilon) = \lim_{m \to \infty} \phi^{(j)}(\varepsilon + \tau_m)$, we obtain as $m \to \infty$ the existence of

$$\lim_{m \to \infty} \frac{\mu_j(\varepsilon + \tau_m) - \mu_j(\varepsilon)}{\tau_m} = \rho_j(\varepsilon) \quad ,$$

and

$$(\psi, A'(\varepsilon)\psi^{(j)}(\varepsilon)) - \rho_j(\varepsilon)(\psi, \psi^{(j)}(\varepsilon)) = 0$$

or

$$P(\varepsilon)A'(\varepsilon)\psi^{(j)}(\varepsilon) = \rho_j(\varepsilon)\psi^{(j)}(\varepsilon) \ , \quad j = p, \ldots, p+h-1 \quad .$$

Thus we have proved (1).

Now the h numbers $\rho_p(\varepsilon), \ldots, \rho_{p+h-1}(\varepsilon)$ are the h eigenvalues of the operator $P(\varepsilon)A'(\varepsilon)$ considered as a Hermitian operator on $\mathfrak{M}$. We shall arrange them in order of magnitude:

$$\pi_p(\varepsilon) \leq \ldots \leq \pi_{p+h-1}(\varepsilon) \quad .$$

Let t_ν be any sequence of positive numbers such that $\lim_{\nu \to \infty} t_\nu = 0$. Then t_ν contains a subsequence $t_{\nu_m} = \tau_m$ such that $\phi^{(j)}(\varepsilon + \tau_m)$, $j = p, \ldots, p+h-1$ converge as $m \to \infty$. Hence, by (1), $\lim_{m \to \infty} \frac{\mu_j(\varepsilon + \tau_m) - \mu_j(\varepsilon)}{\tau_m}$ exist, and these h limits coincide with $\pi_p(\varepsilon), \ldots, \pi_{p+h-1}$. On the other hand, we have

$$\frac{\mu_j(\varepsilon+\tau_m)-\mu_j(\varepsilon)}{\tau_m} = \frac{\mu_j(\varepsilon+\tau_m)-\mu_p(\varepsilon)}{\tau_m} \geq \frac{\mu_k(\varepsilon+\tau_m)-\mu_p(\varepsilon)}{\tau_m}$$

$$\geq \frac{\mu_k(\varepsilon+\tau_m)-\mu_k(\varepsilon)}{\tau_m}$$

for $p \leq k \leq j \leq p+h-1$, and we see that

$$\lim_{m\to\infty} \frac{\mu_j(\varepsilon+\tau_m)-\mu_j(\varepsilon)}{\tau_m} = \pi_j(\varepsilon) \ , \quad j = p,\ldots,p+h-1 \ .$$

Therefore we have proved

$$\lim_{\substack{t\to 0 \\ t>0}} \frac{\mu_j(\varepsilon+t)-\mu_j(\varepsilon)}{t} = r_j(\varepsilon) \quad , \quad j = p,\ldots,p+h-1$$

exist and are a permutation of the h eigenvalues of $P(\varepsilon)A'(\varepsilon)$ on $\mathfrak{M}$. Similarly we can prove the same for

$$\lim_{\substack{t\to 0 \\ t<0}} \frac{\mu_j(\varepsilon+t)-\mu_j(\varepsilon)}{t} = s_j(\varepsilon) \quad , \quad j = p,\ldots,p+h-1 \ .$$

Thus we have proved (2).

Our arguments have shown that there exists an orthonormal set $\omega^{(1)}(\varepsilon),\ldots,\omega^{(n)}(\varepsilon)$ of eigenvectors of $A(\varepsilon)$ such that

$$\pi_j(\varepsilon) = (\omega^{(j)}(\varepsilon), A'(\varepsilon)\omega^{(j)}(\varepsilon)) \quad , \quad j = p,\ldots,p+h-1 \ .$$

Suppose that $\lim_{\nu\to\infty} \varepsilon_\nu = \varepsilon$, and set $t_\nu = \varepsilon_\nu - \varepsilon$. Then there is a subsequence $\tau_m = t_{\nu_m}$ such that $\lim_{m\to\infty} \omega^{(j)}(\varepsilon+\tau_m) = \chi^{(j)}(\varepsilon)$, $j = p,\ldots,p+h-1$. Hence $\chi^{(j)}$ is one of the eigenvectors of $P(\varepsilon)A'(\varepsilon)$ on $\mathfrak{M}$. The sequence $r_j(\varepsilon_\nu)$ contains a subsequence $r_j(\varepsilon+\tau_m)$ which converges to $(\chi^{(j)}(\varepsilon), A'(\varepsilon)\chi^{(j)}(\varepsilon))$ which

is one of the h eigenvalues $r_p(\varepsilon),\ldots,r_{p+h-1}(\varepsilon)$ of $P(\varepsilon)A'(\varepsilon)$ on $\mathfrak{M}$. This establishes (3), and the proof of Theorem 1 is complete.

We shall now prove the

Auxiliary Theorem: *Suppose that $f_1(x),f_2(x),\ldots,f_n(x)$ are n real-valued continuous functions defined in an open interval I such that in I the following conditions hold:*

(1) $f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x)$.

(2) *At each point $f_j(x)$ has a right- and a left-hand derivative, i.e.*

$$r_j(x) = \lim_{\substack{t \to 0 \\ t > 0}} \frac{f_j(x+t) - f_j(x)}{t}, \quad s_j(x) = \lim_{\substack{t \to 0 \\ t < 0}} \frac{f_j(x+t) - f_j(x)}{t}.$$

(3) *If $f_p(x) = \ldots = f_{p+h-1}(x)$ and $f_j(x) \neq f_p(x)$ for j not one of the indices $p,\ldots,p+h-1$, then $r_p(x),\ldots,r_{p+h-1}(x)$ is a permutation of $s_p(x),\ldots,s_{p+h-1}(x)$.*

(4) *If $x_\nu \to x$, with x_ν and x in I, then the set of numbers $r_1(x_\nu),\ldots,r_n(x_\nu)$ has at most the n cluster points $r_1(x),\ldots,r_n(x)$.*

Then there exist n functions $g_1(x),g_2(x),\ldots,g_n(x)$ in I with continuous first derivatives such that at each point x the set of n numbers $g_1(x),\ldots,g_n(x)$ coincides with the set of n numbers $f_1(x),\ldots,f_n(x)$.

Proof: We begin by making some simple remarks.

Remark 1: Suppose that $F(x)$ is a continuous function in $a < x < b$, and such that if x_ν is any sequence with $a < x_\nu < b$ and $x_\nu \rightarrow b$ then $F(x_\nu)$ has at most the cluster points $F_1, \ldots, F_N$. Then $\lim\limits_{\substack{x \rightarrow b \\ a<x<b}} F(x)$ exists (and is one of the numbers $F_1, \ldots, F_N$).

The proof is obvious.

Remark 2: Suppose that $u_j(x)$, $j = 1, \ldots, n$, are n functions defined in a neighborhood of a point y in I, continuous with continuous first derivatives in this neighborhood. Suppose that for each x, $u_1(x), \ldots, u_n(x)$ is a permutation of $f_1(x), \ldots, f_n(x)$. Then there exist two permutations $m_1, \ldots, m_n$ and $n_1, \ldots, n_n$ of the integers $1, \ldots, n$ such that

$$u_j(y) = f_{m_j}(y) \quad , \quad u_j'(y) = r_{m_j}(y) \quad ,$$

and

$$u_j(y) = f_{n_j}(y) \quad , \quad u_j'(y) = s_{n_j}(y) \quad ,$$

for $j = 1, \ldots, n$.

To prove this, we first note that $u_j(y) = f_{a_j}(y)$ for some permutation $a_1, \ldots, a_n$ of $1, \ldots, n$. Now we can choose a sequence $y_\nu > y$ such that $y_\nu \rightarrow y$ and $u_j(y_\nu) = f_{m_j}(y_\nu)$ where the permutation $m_1, \ldots, m_n$ does not depend on ν. Then we have

$$u_j'(y) = \lim_{\nu \rightarrow \infty} \frac{f_{m_j}(y_\nu) - f_{a_j}(y)}{y_\nu - y} \quad .$$

Evidently $\lim\limits_{\nu \rightarrow \infty} f_{m_j}(y_\nu) = f_{a_j}(y)$. Since $f_{m_j}(y)$ is continuous, it follows that $f_{m_j}(y) = f_{a_j}(y)$, and

$$u_j'(y) = \lim_{\nu \rightarrow \infty} \frac{f_{m_j}(y_\nu) - f_{m_j}(y)}{y_\nu - y} = r_{m_j}(y) \quad ,$$

proving Remark 2.

Remark 3: Suppose that $u_j(x)$, $j = 1,\dots,n$, are n functions with continuous first derivatives in $a < x < b$ where a and b are points of I. Suppose that $u_1(x),\dots,u_n(x)$ is a permutation of $f_1(x),\dots,f_n(x)$ for each x in $a < x < b$. Then we have

$$\lim_{\substack{x\to b\\ a<x<b}} u_j(x) = f_{n_j}(b) \quad , \qquad \lim_{\substack{x\to b\\ a<x<b}} u'_j(x) = s_{n_j}(b) \quad ,$$

and an analogous result for the left endpoint a.

To prove this, suppose that $x_\nu \to b$ with $a < x_\nu < b$. Since the set of numbers $u_j(x_\nu)$, $\nu = 1,2,\dots$, can possibly have as cluster points only $f_1(b),\dots,f_n(b)$, it follows from Remark 1 above, that $\lim\limits_{\substack{x\to b\\ a<x<b}} u_j(x)$ exists. We can then choose a sequence $y_\nu \to b$ with $a < y_\nu < b$ and such that $u_j(y_\nu) = f_{n_j}(y_\nu)$ where $n_1,\dots,n_n$ is a permutation of $1,\dots,n$ independent of ν. Hence

$$\lim_{\substack{x\to b\\ a<x<b}} u_j(x) = \lim_{\nu\to\infty} u_j(y_\nu) = \lim_{\nu\to\infty} f_{n_j}(y_\nu) = f_{n_j}(b) \quad .$$

Again suppose that $x_\nu \to b$ with $a < x_\nu < b$. In consequence of Remark 2, the set of numbers $u'_j(x_\nu)$, $j = 1,\dots,n$, is contained in the set $r_j(x_\nu)$, $j = 1,\dots,n$; hence the only possible cluster points of $u'_j(x_\nu)$, $j = 1,\dots,n$ and $\nu = 1,2,\dots$, are the n points $r_1(b),\dots,r_n(b)$. Then, according to Remark 1, $\lim\limits_{\substack{x\to b\\ a<x<b}} u'_j(x)$ exists. For the sequence y_ν selected above, we have $u_j(y_\nu) = f_{n_j}(y_\nu)$, and

$$\lim_{\substack{x\to b\\ a<x<b}} u'_j(x) = \lim_{\nu\to\infty} \frac{f_{n_j}(y_\nu) - f_{n_j}(b)}{y_\nu - b} = s_{n_j}(b) \quad .$$

This completes the proof of Remark 3.

We now commence the proof of our Auxiliary Theorem. Our proof is by complete induction on the number n of functions.

The theorem is true for $n = 1$. In this case $r_1(x) = s_1(x)$ from hypothesis (3), and hence $f_1'(x)$ exists in I. If $x_\nu \longrightarrow x$, with x_ν and x in I, then the numbers $f_1'(x_\nu)$ can have only $r_1(x)$ as a cluster point, by hypothesis (4). Thus $\lim\limits_{x_\nu \to x} f_1'(x_\nu) = f_1'(x)$, and $f_1'(x)$ is continuous in I.

Now we assume the theorem true for $1,\ldots,n-1$, and we shall prove it for n.

First we suppose that K is an open subinterval of I in which there is no point x at which $f_1(x) = \ldots = f_n(x)$. Then suppose that x_o is a point of K such that $f_1(x_o) = f_j(x_o)$ for $1 \leq j \leq m$ and $f_1(x_o) < f_j(x_o)$ for $m < j \leq n$. There exists an open subinterval K_o of K containing x_o such that $f_m(x) < f_j(x)$ for $m < j \leq n$ in K_o. The set of m functions $f_1(x),\ldots,f_m(x)$ and the set of n-m functions $f_{m+1}(x),\ldots,f_n(x)$ each satisfy the hypotheses (in K_o) of the Auxiliary Theorem, it follows from the induction assumption --both m and n-m are less than n--that there exist functions $g_1(x),\ldots,g_m(x)$ and $g_{m+1}(x),\ldots,g_n(x)$ with continuous first derivatives in K_o which coincide with $f_1(x),\ldots,f_m(x)$ and $f_{m+1}(x),\ldots,f_n(x)$, respectively, apart from order. Thus the theorem is proved in K_o.

Let us denote by L the largest subinterval of K which contains K_o such that in L there exist n functions $u_1(x),\ldots,u_n(x)$, with continuous first derivatives, which for each x in L are a permutation of $f_1(x),\ldots,f_n(x)$. We shall prove that L = K. Let b be the right end point of L, and suppose that b were an interior point of K. From Remark 3, we have

$$\lim_{\substack{x \to b \\ a<x<b}} u_j(x) = f_{n_j}(b) \quad , \quad \lim_{\substack{x \to b \\ a<x<b}} u_j'(x) = s_{n_j}(b) \quad .$$

Since b is a point of K, we must have $f_1(b) = f_j(b)$ for $1 \leq j \leq q$ and $f_1(b) < f_j(b)$ for $q < j \leq n$. Hence $f_q(b) < f_j(b)$ for $q < j \leq n$ for some interval $c' < x < c$ containing b. Again using the induction assumption in this interval, there exist functions $v_1(x),\ldots,v_n(x)$ with continuous first derivatives in $c' < x < c$ which for each x are a permutation of $f_1(x),\ldots,f_n(x)$. According to Remark 2, we have $v_j(b) = f_{k_j}(b)$ and $v_j'(b) = s_{k_j}(b)$. Now let us define $w_1(x),\ldots,w_n(x)$ as that permutation of $v_1(x),\ldots,v_n(x)$ such that $w_j(b) = f_{n_j}(b) = \lim\limits_{\substack{x \to b \\ a<x<b}} u_j(x)$ and

$w_j'(b) = s_{n_j} = \lim\limits_{\substack{x \to b \\ a<x<b}} u_j'(x)$. Finally define

$$g_j(x) = \begin{cases} u_j(x) & \text{for } x \text{ in } L \ (a < x < b) , \\ w_j(x) & \text{for } b \leq x < c \quad . \end{cases}$$

Obviously the functions $g_j(x)$ satisfy our theorem in the larger interval $a < x < c$. This is a contradiction. It follows that $L = K$, and our theorem is true in K.

Let us denote by I' the subset of I consisting of those points x where either $f_1(x) = \ldots = f_n(x)$ or $r_1(x) = \ldots = r_n(x)$ is not the case. The set I' is open, and hence the union of non-overlapping open intervals I_ν, $\nu = 1,2,\ldots$. It is easy to see that I_ν does not contain any cluster point of the set of points where $f_1(x) = \ldots = f_n(x)$. Thus if there are infinitely many points in I_ν where $f_1(x) = \ldots = f_n(x)$, there can only be a countable number and these can only cluster at the endpoints of I_ν. We denote these points by $\alpha_1,\alpha_2,\ldots$ and $\beta_1,\beta_2,\ldots$ where

$$\alpha \leq \ldots < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < \ldots \leq \beta$$

and $\alpha < x < \beta$ is the interval I_ν.

In $\alpha_1 < x < \beta_1$ we can find functions $u_1(x), \ldots, u_n(x)$, with continuous first derivatives, which coincide with $f_1(x), \ldots, f_n(x)$ apart from order. We can do the same in $\beta_1 < x < \beta_2$; we call the functions $v_1(x), \ldots, v_n(x)$. From Remark 3, it follows that

$$\lim_{\substack{x \to \beta_1 \\ \alpha_1 < x < \beta_1}} u_j(x) = f_{n_j}(\beta_1) \quad , \qquad \lim_{\substack{x \to \beta_1 \\ \alpha_1 < x < \beta_1}} u'_j(x) = s_{n_j}(\beta_1) \quad ,$$

and

$$\lim_{\substack{x \to \beta_1 \\ \beta_1 < x < \beta_2}} v_j(x) = f_{m_j}(\beta_1) \quad , \qquad \lim_{\substack{x \to \beta_1 \\ \beta_1 < x < \beta_2}} v'_j(x) = r_{m_j}(\beta_1) \quad .$$

Hence there is a permutation $w_1(x), \ldots, w_n(x)$ of $v_1(x), \ldots, v_n(x)$ such that $\lim\limits_{x \to \beta_1 -} u_j(x) = \lim\limits_{x \to \beta_1 +} w_j(x)$ and $\lim\limits_{x \to \beta_1 -} u'_j(x) = \lim\limits_{x \to \beta_1 +} w'_j(x)$. Therefore, if we define in $\beta_1 < x < \beta_2$, $u_j(x) = w_j(x)$, and $u_j(\beta_1) = \lim\limits_{x \to \beta_1 -} u_j(x)$, then the $u_j(x)$ has continuous first derivatives in $\alpha_1 < x < \beta_2$, and are a permutation of the $f_j(x)$. Continuing in this way, we can define $u_1(x), \ldots, u_n(x)$ functions with continuous first derivatives in the whole interval I_ν such that at each point x, $u_1(x), \ldots, u_n(x)$ is a permutation of $f_1(x), \ldots, f_n(x)$. Thus we have succeeded in defining $u_1(x), \ldots, u_j(x)$ in I' so that $u_1(x), \ldots, u_n(x)$ have continuous first derivatives in I' and $u_1(x), \ldots, u_n(x)$ is a permutation of $f_1(x), \ldots, f_n(x)$ at each point x of I'.

Finally we define $g_j(x) = u_j(x)$ for x in I' and $g_j(x) = f_j(x)$ for x in $I - I'$. We claim that $g_j(x)$ is a continuous function with continuous first derivatives in I. Recall that $I - I'$ consists of those points x in I such that

$$f_1(x) = \ldots = f_n(x) \quad , \quad r_1(x) = \ldots = r_n(x) \quad .$$

We need only verify our assertion in $I - I'$ as it is obviously true in I'. First note that $g_j'(x)$ exists because $f_1(x) = \ldots = f_n(x)$ and $r_1(x) = \ldots = r_n(x)$, whence $s_1(x) = \ldots = s_n(x) = r_1(x)$. Therefore $r_j(x) = s_j(x)$, and so $g_j'(x) = f_j'(x)$ exists. We next must prove that $g_j'(x)$ is continuous at every point x in $I - I'$. Suppose that x_ν is a sequence of points in $I - I'$ which tend to x, a point in $I - I'$. Then $g_j'(x_\nu) = f_j'(x_\nu) = r_j(x_\nu)$. Since the only cluster points of $r_1(x_\nu),\ldots,r_n(x_\nu)$ are $r_1(x) = \ldots = r_n(x)$, it follows that $\lim_{\nu \to \infty} g_j'(x_\nu) = r_j(x) = g_j'(x)$. Suppose that x_ν is a sequence of points in I' which tend to x, a point in $I - I'$. Since $g_j'(x_\nu) = r_j(x_\nu)$, whose only possible cluster point is $r_1(x) = \ldots = r_n(x)$, it follows that $\lim_{\nu \to \infty} g_j'(x_\nu) = g_j'(x)$. This completes the proof of the Auxiliary Theorem.

Theorem 1 together with the Auxiliary Theorem prove our main Theorem (see pp. 57-58).

Chapter II. Perturbation of the Discrete Spectrum

§1. Power series in Hilbert Space.

We shall now consider eigenvalue problems in a Hilbert space[*] $\mathfrak{H}$ of infinite dimension. $\mathfrak{H}$ is a linear space equipped with an inner product, a complex-valued functional (written (u,v)) defined for every pair of elements u, v in $\mathfrak{H}$ with the following properties:

(a) $(u,v) = \overline{(v,u)}$.

(b) (u,v) is for fixed u a linear functional of v, i.e. $(u,av+a'v') = a(u,v) + a'(u,v')$ for any elements u, v, v' and complex numbers a, a'.

(c) $(u,u) \geq 0$, and $(u,u) = 0$ only if $u = 0$.

The norm of u (written $\|u\|$) is defined by

$$\|u\| = (u,u)^{\frac{1}{2}} .$$

Important inequalities are the triangle inequality,

$$\|u+v\| \leq \|u\| + \|v\| ,$$

and Schwarz's inequality,

$$|(u,v)| \leq \|u\|\cdot\|v\| .$$

The Hilbert space is assumed to be complete. If (v_n) is any sequence of elements in $\mathfrak{H}$ such that

$$\|v_n - v_m\| \to 0 \quad \text{as} \quad n,m \to \infty$$

then there exists an element v in $\mathfrak{H}$ such that

$$\|v_n - v\| \to 0 \quad \text{as} \quad n \to \infty .$$

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[*] For the elements of Hilbert space theory we refer to the standard texts.

Example. The set of all complex-valued functions $u(x,y,z)$ defined in Euclidean x,y,z-space which are square integrable,

$$\iiint_{-\infty}^{\infty} |u(x,y,z)|^2 dxdydz < \infty$$

is a Hilbert space, under the inner product

$$(v,u) = \iiint_{-\infty}^{\infty} \overline{v(x,y,z)}u(x,y,z)dxdydz \quad .$$

Evidently

$$\|u\| = \left(\iiint_{-\infty}^{\infty} |u(x,y,z)|^2 dxdydz\right)^{\frac{1}{2}} \quad .$$

An element $u(\varepsilon)$ of $\mathfrak{H}$ defined for all (real or complex) numbers ε with $|\varepsilon| < \rho$ is said to be a power series in ε convergent for $|\varepsilon| < \rho$ when there is a sequence of elements $u_o, u_1, u_2, \ldots$ in $\mathfrak{H}$, independent of ε, such that

$$u(\varepsilon) = u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots , \quad \text{for } |\varepsilon| < \rho \quad .$$

More explicitly, the meaning of the infinite series is that

$$\lim_{n\to\infty} \|u(\varepsilon) - (u_o + \varepsilon u_1 + \ldots + \varepsilon^n u_n)\| = 0 , \text{ for } |\varepsilon| < \rho \quad .$$

Occasionally we shall confine our attention to real ε. It is important to note that convergence of the power series is with respect to the metric of the Hilbert space. To illustrate this point, consider the function $u = e^{-\varepsilon r}$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, and ε is a small positive number. (We recall that this function $u(\varepsilon)$ is an (unnormalized) eigenfunction of Schrödinger's equation

$$-(u_{xx} + u_{yy} + u_{zz}) - \frac{2Z}{r} u = \lambda u \quad , \qquad -\infty < x,y,z < +\infty$$

belonging to the eigenvalue $\lambda = -Z^2$, where we have put $Z = \varepsilon$.) Although

$$u(\varepsilon) = e^{-\varepsilon r} = 1 - \varepsilon r + \frac{1}{2!}\varepsilon^2 r^2 - \ldots$$

is evidently a convergent power series in the ordinary sense for all ε at a fixed point (x,y,z), the question at hand is whether or not $u(\varepsilon)$ is a power series in the perturbation parameter ε in a neighborhood of $\varepsilon = 0$ with respect to the norm $\|u\| = \left(\iiint_{-\infty}^{\infty} |u|^2 dxdydz\right)^{\frac{1}{2}}$. The answer is plainly no. Indeed for $\varepsilon = 0$, the function $u(\varepsilon) = e^{-\varepsilon r}$ does not even belong to the Hilbert space, i.e. the function $u(0) \equiv 1$ does not have a finite square integral over the whole x,y,z-space. Whereas the eigenvalue $\lambda = -\varepsilon^2$ is a convergent power series, the corresponding eigenfunction $u = e^{-\varepsilon r}$ is not a convergent power series in Hilbert space for small $|\varepsilon|$.

Certain well-known concepts and theorems concerning ordinary power series can be extended to power series in Hilbert space. For example, every power series in Hilbert space has a radius R of convergence $(0 \leq R \leq \infty)$, i.e. the power series converges for every complex number ε when $|\varepsilon| < R$, and diverges for $|\varepsilon| \geq R$. In fact, R can be computed by the Cauchy-Hadamard formula for the radius of convergence of an ordinary power series; if the power series in Hilbert space is $u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$, then its radius of convergence is given by

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|u_n\|}} \quad .$$

A linear operator A defined on $\mathfrak{H}$ is said to be bounded if there exists a number a such that

$$\|Au\| \leq a\|u\| \quad \text{for all } u \text{ in } \mathfrak{H} .$$

The smallest number a for which this inequality is true is called the bound of A and is denoted by $\|A\|$. Evidently the bound of A is given by

$$\|A\| = \underset{u \neq 0}{\text{l.u.b.}} \frac{\|Au\|}{\|u\|} \quad .$$

Suppose that, for each ε with $|\varepsilon| < \rho$. the operator $A(\varepsilon)$ is bounded. $A(\varepsilon)$ is said to be a regular operator for $|\varepsilon| < \rho$ if, for each element u of $\mathfrak{H}$, $A(\varepsilon)u$ is a power series in $\mathfrak{H}$. It can be proved[*], using a theorem of Banach, that an equivalent definition is: there exist a sequence of bounded operators $A_0, A_1, \ldots$ independent of ε and a positive number c such that for each element u in $\mathfrak{H}$

(1) $A(\varepsilon)u = A_0u + \varepsilon A_1u + \varepsilon^2 A_2u + \ldots$ is a convergent power series for $|\varepsilon| < \rho$,

(2) $\|A_nu\| \leq c^{n+1}\|u\|$, $n = 0,1,2,\ldots$.

Suppose that $A_0, A_1, A_2, \ldots$ is any sequence of bounded operators, and $a_0, a_1, a_2, \ldots$ is a sequence of numbers such that

$$\|A_n\| \leq a_n \quad , \qquad n = 0,1,2,\ldots \quad .$$

Then the ordinary power series

$$a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots$$

is said to be a majorant of the power series

$$A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots \quad .$$

It is obvious that if the majorant series converges for $|\varepsilon| < \rho$, then

$$A_0u + \varepsilon A_1u + \varepsilon^2 A_2u + \ldots$$

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[*] F. Rellich, Störungstheorie der Spectralzerlegung I., Math. Ann. vol. 113, p. 600-619 (1936), especially p. 607 ff.

is a convergent power series in $\mathfrak{H}$ for $|\varepsilon| < \rho$. The above definition and statement can plainly be extended to power series in several variables.

If an operator A is not defined on the whole Hilbert space $\mathfrak{H}$ but only on a dense subspace $\mathfrak{D}_A$, it is said to be bounded if there is a number a such that $\|Au\| \leq a\|u\|$ for all u in $\mathfrak{D}_A$. In this case A can be extended to an operator on $\mathfrak{H}$ with the same bound, and the extension is unique.

A most important class of bounded operators is that of integral operators. Suppose that k(x,y) is a continuous function of x,y for $0 \leq x,y \leq 1$; then we define the integral operator

$$Ku = \int_0^1 k(x,y)u(y)dy$$

for all functions u(x) continuous in $0 \leq x \leq 1$. The appropriate Hilbert space $\mathfrak{H}$ in which to study the operator K is the space of all complex-valued functions u(x) defined for $0 \leq x \leq 1$ which are quadratically integrable, $\int_0^1 |u(x)|^2 dx < \infty$. The inner product is

$$(v,u) = \int_0^1 \overline{v(x)}u(x)dx \quad .$$

We thus have defined the operator K on the dense subspace $\mathfrak{C}$ of continuous functions of $\mathfrak{H}$. Of course K is bounded; and the domain of definition of K, which was originally $\mathfrak{C}$, can be enlarged to $\mathfrak{H}$ in an obvious way.

In addition to the operator K we consider a family of integral operators

$$K(\varepsilon)u = \int_0^1 k(x,y;\varepsilon)u(y)dy$$

where

$$k(x,y;\varepsilon) = k(x,y) + \varepsilon k_1(x,y) + \varepsilon^2 k_2(x,y) + \dots$$

is a convergent power series (in the ordinary sense) for $|\varepsilon| < \rho$ with the convergence uniform in $0 \leq x,y \leq 1$, and where $k_1(x,y), k_2(x,y), \dots$ are all continuous functions of x,y in $0 \leq x,y \leq 1$. Now the integral operator $K(\varepsilon)$ may be regarded as defined on $\mathfrak{C}$ or on $\mathfrak{H}$. In the first case we can extend $\bar{K}(\varepsilon)$ to an operator $K(\varepsilon)$ on $\mathfrak{H}$, and we suppose this done. Is $K(\varepsilon)$ a convergent power series in ε for small $|\varepsilon|$? To show that it is, we define the sequence of integral operators

$$A_n u = \int_0^1 k_n(x,y)u(y)dy \quad , \qquad n = 0,1,2,\dots$$

where $k_0(x,y) = k(x,y)$. It follows from our assumptions about $k(x,y;\varepsilon)$ that there is a number c such that

$$|k_n(x,y)| \leq c^{n+1} \quad , \qquad n = 0,1,2,\dots \quad ,$$

and consequently

$$\|A_n u\| \leq c^{n+1} \|u\| \quad , \qquad n = 0,1,2,\dots$$

for all u in $\mathfrak{H}$. Therefore

$$K(\varepsilon)u = A_0 u + \varepsilon A_1 u + \varepsilon^2 A_2 u + \dots$$

for $|\varepsilon| < \frac{1}{c}$, and $K(\varepsilon)$ is a power series in ε for $|\varepsilon| < \frac{1}{c}$.

If we make the additional assumption that $k(x,y;\varepsilon) = \overline{k(y,x;\varepsilon)}$ for all real ε with $|\varepsilon| < \rho$, then the character of the eigenvalue problem

$$\int_0^1 k(x,y;\varepsilon)\phi(y)dy = \lambda\phi(x)$$

is well known. Every eigenvalue λ different from zero has finite multiplicity. The eigenvalues λ and the eigenfunctions

depend on ε. Are they convergent power series in ε for small $|\varepsilon|$? We answer this question in a much more general case, replacing the integral $K(\varepsilon)$ by an arbitrary bounded Hermitian operator $A(\varepsilon)$.

§2. Perturbation of an isolated eigenvalue of finite multiplicity of a bounded operator.

Theorem 1: *Suppose that $A(\varepsilon)$ is a bounded operator defined on $\mathfrak{H}$ which is a power series in ε*

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots$$

convergent in a neighborhood of $\varepsilon = 0$. Suppose that for real ε with ε small, and any elements u, v in $\mathfrak{H}$ we have $(u, A(\varepsilon)v) = (A(\varepsilon)u, v)$. Suppose that λ is an eigenvalue of finite multiplicity h of the operator $A(0) = A_0 = A$, and suppose that there are positive numbers d_1 and d_2 such that the spectrum of A in the open interval $\lambda - d_1 < \mu < \lambda + d_2$ consists exactly of the point eigenvalue λ. Then there exist ordinary power series

$$\lambda_1(\varepsilon), \dots, \lambda_h(\varepsilon) \quad ,$$

and power series in Hilbert space

$$\phi^{(1)}(\varepsilon), \dots, \phi^{(h)}(\varepsilon) \quad ,$$

all convergent in a neighborhood of $\varepsilon = 0$, which satisfy the following conditions:

(1) *The element $\phi^{(i)}(\varepsilon)$ is an eigenelement of $A(\varepsilon)$ belonging to the eigenvalue $\lambda_i(\varepsilon)$, i.e.*

$$A(\varepsilon)\phi^{(i)}(\varepsilon) = \lambda_i(\varepsilon)\phi^{(i)}(\varepsilon) \quad , \quad i = 1, \dots, h \quad .$$

Furthermore $\lambda_i(0) = \lambda$, $i = 1,\ldots,h$; *and for real* ε *the eigenelements form an orthonormal set, i.e. for real* ε,

$$(\phi^{(i)}(\varepsilon),\phi^{(j)}(\varepsilon)) = \delta_{ij} \quad , \qquad i,j = 1,\ldots,h \quad .$$

(2) *For each pair of positive numbers* d_1', d_2' *with* $d_1' < d_1$, $d_2' < d_2$ *there exists a positive number* ρ *such that the spectrum of* $A(\varepsilon)$ *in* $\lambda - d_1' \leq \mu \leq \lambda + d_2'$ *for real* ε *with* $|\varepsilon| < \rho$ *consists exactly of the points* $\lambda_1(\varepsilon),\ldots,\lambda_h(\varepsilon)$.

Remark 1. The proof we shall give below rests on the fact that, as in the finite dimensional case, the problem for the eigenvalues can be treated separately. We can derive an equation for the eigenvalues which is independent of the eigenelements.

Remark 2. Consider the equation

$$(A - \lambda)u = v$$

where v is a given element of $\mathfrak{H}$. Since λ is an eigenvalue of multiplicity h of the Hermitian operator $A = A(0)$, the homogeneous equation

$$(A - \lambda)u = 0$$

has h linearly independent solutions $\phi^{(1)},\ldots,\phi^{(h)}$ which we can assume to be orthonormal

$$(\phi^{(i)},\phi^{(k)}) = \delta_{ik} \quad , \qquad i,k = 1,\ldots,h \quad .$$

Of course the operator $A - \lambda$ has no inverse, but there is a (uniquely determined) bounded Hermitian operator R such that

(1) $R\phi^{(i)} = 0$, $i = 1,\ldots,h$,

(2) $R(A - \lambda)f = f - \sum_{i=1}^{h} (\phi^{(i)},f)\phi^{(i)}$, for every f in $\mathfrak{H}$.

If we introduce the projector P as

$$Pf = \sum_{i=1}^{h} (\phi^{(i)},f)\phi^{(i)} \quad ,$$

we can rewrite the defining properties of R as

(1) $RP = 0$,

(2) $R(A-\lambda) = 1-P$.

R is called the pseudo-inverse of $A-\lambda$.

If we were in the n-dimensional space $\mathfrak{H}^n$, then we could complete $\phi^{(1)},\ldots,\phi^{(h)}$ with eigenvectors $\phi^{(h+1)},\ldots,\phi^{(n)}$ belonging to the eigenvalues $\lambda_{h+1},\ldots,\lambda_n$ respectively in such a way that $(\phi^{(i)},\phi^{(k)}) = \delta_{ik}$, $i,k = 1,\ldots,n$. Expanding u and v as

$$u = \sum_{i=1}^{n} (\phi^{(i)},u)\phi^{(i)} \quad , \qquad v = \sum_{i=1}^{n} (\phi^{(i)},v)\phi^{(i)} \quad ,$$

and inserting u and v into the equation

$$(A-\lambda)u = v \quad ,$$

we obtain

$$(\phi^{(i)},v) = 0 \quad , \qquad i = 1,\ldots,h \quad ,$$

and

$$(\phi^{(i)},v) = (\lambda_i-\lambda)(\phi^{(i)},u) \quad , \qquad i = h+1,\ldots,n \quad .$$

Thus if the equation for u has a solution it is necessary for v to be orthogonal to all solutions of the homogeneous equation. Assuming this to be the case, we set

$$u = \sum_{i=1}^{h} u_i\phi^{(i)} + \sum_{\lambda_i\neq\lambda} \frac{(\phi^{(i)},v)}{\lambda_i-\lambda}\,\phi^{(i)} \quad ,$$

where the u_i are arbitrary numbers; and we observe that u thus defined is the complete solution of the equation. Let us denote by P the projector into the space spanned by

$\phi^{(1)},\ldots,\phi^{(h)}$, and by P_i the projector into the one-dimensional space spanned by $\phi^{(i)}$, $i = h+1,\ldots,n$. Finally we define

$$R = \sum_{\lambda_i \neq \lambda} \frac{P_i}{\lambda_i - \lambda} \quad ,$$

and we see that R is the pseudo-inverse of $A - \lambda$.

In the general case of a Hilbert space $\mathfrak{H}$ of infinite dimension we must use the resolution of the identity associated with the bounded Hermitian operator A. Suppose that

$$A = \int_{-\infty}^{\infty} \nu \, dE_\nu \quad ;$$

then the pseudo-inverse of $A - \lambda$ can be written as

$$R = \int_{-\infty}^{\lambda - \frac{d_1}{2}} \frac{1}{\nu - \lambda} \, dE_\nu + \int_{\lambda + \frac{d_2}{2}}^{\infty} \frac{1}{\nu - \lambda} \, dE_\nu \quad .$$

Remark 3. To derive the equation satisfied by the perturbed eigenvalues, we suppose that the perturbed eigenvalue $\lambda(\varepsilon)$ and corresponding eigenfunction $\phi(\varepsilon)$ exist so that $A(\varepsilon)\phi(\varepsilon) = \lambda(\varepsilon)\phi(\varepsilon)$. We put

$$B(\varepsilon) = A(\varepsilon) - A \ , \quad \mu(\varepsilon) = \lambda(\varepsilon) - \lambda \quad ,$$

to find that

$$(A - \lambda)\phi(\varepsilon) = (\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon) \quad .$$

From this equation, it follows that

$$(\phi^{(i)},(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon)) = 0 \quad , \qquad i = 1,\ldots,h \quad ,$$

or

$$((\mu(\varepsilon) - B(\varepsilon))\phi^{(i)},\phi(\varepsilon)) = 0 \quad , \qquad i = 1,\ldots,h \quad .$$

Furthermore we have

$$\phi(\varepsilon) = \psi + R(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon) \quad ,$$

where $\psi = \sum_{k=1}^{h} c_k \phi^{(k)}$, and R is the pseudo-inverse of $A - \lambda$. Let us write

$$S = R(\mu(\varepsilon) - B(\varepsilon)) \quad .$$

Without worrying about convergence now, the equation

$$\phi(\varepsilon) = \psi + S\phi(\varepsilon)$$

can be solved for $\phi(\varepsilon)$ by the Neumann series

$$\phi(\varepsilon) = \sum_{\nu=0}^{\infty} S^{\nu}\psi \quad .$$

With this expression for $\phi(\varepsilon)$ we find

$$\Big((\mu(\varepsilon) - B(\varepsilon))\phi^{(i)}, \sum_{\nu=0}^{\infty} S^{\nu}\psi\Big) = 0 \ , \quad i = 1,\ldots,h \quad ,$$

or

$$\sum_{k=1}^{h} \Big((\mu(\varepsilon) - B(\varepsilon))\phi^{(i)}, \sum_{\nu=0}^{\infty} S^{\nu}\phi^{(k)}\Big)c_k = 0 \ , \quad i = 1,\ldots,h \quad ,$$

from which it follows that

$$\det \Big(((u(\varepsilon) - B(\varepsilon))\phi^{(i)}, \sum_{\nu=0}^{\infty} S^{\nu}\phi^{(k)})\Big) = 0 \quad .$$

This equation does not involve the perturbed eigenvector $\phi(\varepsilon)$ and must be satisfied (subject to convergence considerations) by the perturbed eigenvalue $\lambda(\varepsilon) = \lambda + \mu(\varepsilon)$.

This is the starting point for the proof of our Theorem 1.

Proof: We define $f_{ik}(\mu,\varepsilon)$ by the relation

$$f_{ik}(\mu,\varepsilon) = (\phi^{(i)}, \sum_{\nu=0}^{\infty} (\mu - B(\varepsilon))[R(\mu - B(\varepsilon))]^{\nu} \phi^{(k)}) \quad ,$$

$$i,k = 1,\ldots,h \quad .$$

We know that $B(\varepsilon) = \varepsilon A_1 + \varepsilon^2 A_2 + \dots$ has the majorant $\varepsilon c^2 + \varepsilon^2 c^3 + \dots$. If $\|R\| \leq r$ then $R(\mu - B(\varepsilon))$ has the majorant $r(\mu + \varepsilon c^2 + \varepsilon^2 c^3 + \dots) = r\left(\mu + \frac{\varepsilon c^2}{1 - \varepsilon c}\right)$ if $|\varepsilon c| \leq \frac{1}{2}$. Hence $\sum_{\nu=0}^{\infty} [R(\mu - B(\varepsilon))]^{\nu}$ has the majorant $\left(1 - r(\mu + \frac{\varepsilon c^2}{1 - \varepsilon c})\right)^{-1}$ provided $|\mu|$ and $|\varepsilon|$ are so small that $\left|r\left(\mu + \frac{\varepsilon c^2}{1 - \varepsilon c}\right)\right| \leq \frac{1}{2}$. Therefore $\sum_{\nu=0}^{\infty} (\mu - B(\varepsilon))[R(\mu - B(\varepsilon))]^{\nu}$ is a convergent power series in μ and ε convergent in a neighborhood of $\mu = \varepsilon = 0$; and therefore the same holds for

$$F(\mu,\varepsilon) = \det (f_{ik}(\varepsilon)) \quad .$$

For real μ and ε we observe that

$$(\phi^{(i)},(\mu - B(\varepsilon))[R(\mu - B(\varepsilon))]^{\nu}\phi^{(k)})$$

$$= ([(\mu - B(\varepsilon))R]^{\nu}(\mu - B(\varepsilon))\phi^{(i)},\phi^{(k)})$$

$$= ((\mu - B(\varepsilon))[R(\mu - B(\varepsilon))]^{\nu}\phi^{(i)},\phi^{(k)}) \quad ,$$

and hence $f_{ik}(\mu,\varepsilon) = \overline{f_{ki}(\mu,\varepsilon)}$ for real μ, ε.

Now we see that

$$f_{ik}(\mu,0) = (\phi^{(i)}, \sum_{\nu=0}^{\infty} \mu^{\nu+1}R^{\nu}\phi^{(k)}) = (\phi^{(i)},\mu\phi^{(k)}) = \mu\delta_{ik} \quad ,$$

and hence $F(\mu,0) = \det (\mu\delta_{ik}) = \mu^h$.

If for small $|\varepsilon|$ there exists a complex-valued function $\mu(\varepsilon)$ with $\lim_{\varepsilon \to 0} \mu(\varepsilon) = 0$ and $F(\mu(\varepsilon),\varepsilon) = 0$, then we can find h functions $c_1(\varepsilon),\dots,c_h(\varepsilon)$ for which

$$\sum_{k=1}^{h} f_{ik}(\mu(\varepsilon),\varepsilon)c_k(\varepsilon) = 0$$

and

$$\sum_{k=1}^{h} |c_k(\varepsilon)|^2 = 1 \quad .$$

Then we define

$$\psi(\varepsilon) = \sum_{k=1}^{h} c_k(\varepsilon)\phi^{(k)} \quad \text{and} \quad \phi(\varepsilon) = \sum_{\nu=0}^{\infty} [R(\mu(\varepsilon) - B(\varepsilon))]^{\nu}\psi(\varepsilon) \quad ,$$

at least for $|\varepsilon|$ small. Since

$$\phi(\varepsilon) = \psi(\varepsilon) + R(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon) \quad ,$$

we obtain

$$\begin{aligned}(A_o-\lambda)\phi(\varepsilon) &= (A_o-\lambda)R(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon) \\ &= (\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon) - \sum_{i=1}^{h} (\phi^{(i)},(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon))\phi^{(i)} \quad .\end{aligned}$$

But we know that

$$\begin{aligned}&(\phi^{(i)},(\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon)) \\ &\quad = \sum_{k=1}^{h} (\phi^{(i)}, \sum_{\nu=0}^{\infty} (\mu - B(\varepsilon))[R(\mu - B(\varepsilon))]^{\nu}\phi^{(k)})c_k(\varepsilon) \\ &\quad = \sum_{k=1}^{h} f_{ik}(\mu(\varepsilon),\varepsilon)c_k(\varepsilon) = 0 \quad , \qquad i = 1,\ldots,h \quad ;\end{aligned}$$

and hence

$$(A_o-\lambda)\phi(\varepsilon) = (\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon)$$

or

$$A(\varepsilon)\phi(\varepsilon) = (\lambda + \mu(\varepsilon))\phi(\varepsilon) \quad .$$

Since $\phi(\varepsilon) = \psi(\varepsilon) + \sum_{\nu=1}^{\infty} [R(\mu(\varepsilon) - B(\varepsilon))]^{\nu}\psi(\varepsilon)$, it is easy to see that $\|\phi(\varepsilon) - \psi(\varepsilon)\| < \frac{1}{2}$ for $|\varepsilon|$ small. From this observation and the fact that $\|\psi(\varepsilon)\| = 1$, we note that $\phi(\varepsilon) \neq 0$. Hence $\phi(\varepsilon)$ is an eigenelement belonging to the eigenvalues $\lambda + \mu(\varepsilon)$ for small $|\varepsilon|$.

It follows from the Weierstrass preparation theorem that

$$F(\mu,\varepsilon) = (\mu^h + p_1(\varepsilon)\mu^{h-1} + \dots + p_h(\varepsilon))E(\mu,\varepsilon)$$

where the power series $p_1(\varepsilon),\dots,p_h(\varepsilon)$ converge for small $|\varepsilon|$, and $E(\mu,\varepsilon)$ is a power series convergent for small $|\mu|$, $|\varepsilon|$ with $E(0,0) = 1$. There are h functions $\mu_1(\varepsilon),\dots,\mu_h(\varepsilon)$, defined in a neighborhood of $\varepsilon = 0$, such that

$$\mu^h + p_1(\varepsilon)\mu^{h-1} + \dots + p_h(\varepsilon) = \prod_{j=1}^{h} (\mu - \mu_j(\varepsilon))$$

and

$$\lim_{\varepsilon \to 0} \mu_j(\varepsilon) = 0 \quad , \qquad j = 1,\dots,h \quad .$$

Now $\mu_j(\varepsilon)$ is real for real ε (with $|\varepsilon|$ small) because $\lambda + \mu_j(\varepsilon)$ is an eigenvalue of $A(\varepsilon)$ if $|\varepsilon|$ is small enough. Since the operator $A(\varepsilon)$ for ε real has only real eigenvalues, it follows that $\mu_j(\varepsilon)$ is real for real ε. Since the $\mu_j(\varepsilon)$ are real for real ε, we can suppose that $\mu_j(\varepsilon)$, $j = 1,\dots,h$, are power series convergent in a neighborhood of $\varepsilon = 0$. We select one of the $\mu_j(\varepsilon)$ and denote it by $\mu(\varepsilon)$. Moreover we can now take solutions $c_1(\varepsilon),\dots,c_h(\varepsilon)$ of the h homogeneous equations

$$\sum_{k=1}^{h} f_{ik}(\mu(\varepsilon),\varepsilon)c_k(\varepsilon) = 0 \ , \quad i = 1,\dots,h \quad ,$$

to be power series in ε convergent near $\varepsilon = 0$; and for real ε

$$\sum_{k=1}^{h} |c_k(\varepsilon)|^2 = 1 \quad .$$

This is possible because the $f_{ik}(\mu(\varepsilon),\varepsilon)$ are power series in ε. Furthermore the elements $\phi(\varepsilon)$, defined above, are also power series in ε. This establishes the existence of an eigenvalue $\lambda(\varepsilon)$ and a corresponding eigenelement $\phi(\varepsilon)$ which are power series in ε for small $|\varepsilon|$, and such that $\lambda(0) = \lambda$.

Our theorem claims there are h such power series. Their existence is easily proved (in a way similar to the corresponding situation in Chapter I, §1, pp. 43-44) by complete induction on the multiplicity h. We shall not give the proof.

We now prove the last part of our theorem. We define the projector $P(\varepsilon)$ by

$$P(\varepsilon)u = \sum_{i=1}^{h} (\phi^{(i)}(\varepsilon),u)\phi^{(i)}(\varepsilon) \quad ,$$

and the operator $D(\varepsilon)$ by

$$D(\varepsilon) = A(\varepsilon) - (d_1 + d_2)P(\varepsilon) \quad .$$

If $u(\varepsilon)$ is an element of $\mathfrak{H}$ orthogonal to the space spanned by $\phi^{(1)}(\varepsilon),\ldots,\phi^{(h)}(\varepsilon)$, then $D(\varepsilon)u = A(\varepsilon)u$. Furthermore we note that $D(\varepsilon)\phi^{(k)}(\varepsilon) = (\lambda(\varepsilon) - (d_1 + d_2))\phi^{(k)}(\varepsilon)$ for real ε. Thus for real ε the spectrum of $D(\varepsilon)$ is obtained from that of $A(\varepsilon)$ by shifting the eigenvalues $\lambda_k(\varepsilon)$ of $A(\varepsilon)$ to $\lambda_k(\varepsilon) - d_1 - d_2$, and leaving the other points of the spectrum of $A(\varepsilon)$ where they are. The spectrum of $D(0)$ is empty in $\lambda - d_1 < \mu < \lambda + d_1$. We must prove that to each pair d_1', d_2' of numbers with $0 < d_1' < d_1$ and $0 < d_2' < d_2$, there is a positive number ρ such that the spectrum of $D(\varepsilon)$ is empty in $\lambda - d_1' \leq \mu \leq \lambda + d_2'$ for real ε with $|\varepsilon| < \rho$. Let us put $\lambda' = \lambda + \frac{1}{2}(d_2' - d_1')$. Since no point in the spectrum of $D(0)$ lies in the interval

$$\lambda' - \left(\frac{d_1' + d_2'}{2} + d_1 - d_1'\right) < \mu < \lambda' + \left(\frac{d_1' + d_2'}{2} + d_2 - d_2'\right) \quad ,$$

we find that $\|D(0) - \lambda')u\| \geq \delta\|u\|$, where

$$\delta = \min\left(\frac{d_1' + d_2'}{2} + d_1 - d_1', \frac{d_1' + d_2'}{2} + d_2 - d_2'\right) > \frac{d_1' + d_2'}{2} \quad .$$

Since $D(\varepsilon)$ is a power series convergent for small $|\varepsilon|$, we can find a positive number ρ such that $\|(D(\varepsilon)-\lambda')u\| > \frac{d_1' + d_2'}{2}\|u\|$ for all u of $\mathfrak{H}$ for $|\varepsilon| < \rho$. But this implies that the interval $\lambda' - \frac{d_1'+d_2'}{2} \leq \mu \leq \lambda' + \frac{d_1'+d_2'}{2}$, or what is the same, $\lambda - d_1' \leq \mu \leq \lambda + d_2'$, contains no point of the spectrum of $D(\varepsilon)$. Therefore the spectrum of $A(\varepsilon)$ for ε real and $|\varepsilon| < \rho$ has only the h eigenvalues $\lambda_1(\varepsilon),\ldots,\lambda_h(\varepsilon)$ in the interval $\lambda - d_1' \leq \mu \leq \lambda + d_2'$.

§3. Essentially self-adjoint and self-adjoint operators.

A linear operator A defined on a subspace $\mathfrak{A}$ (the "domain" of A) of a Hilbert space $\mathfrak{H}$ is said to be a Hermitian operator if (1) $\mathfrak{A}$ is dense in $\mathfrak{H}$, and (2) $(v,Au) = (Av,u)$ for all u, v in $\mathfrak{A}$.

A Hermitian operator A on $\mathfrak{A}$ is said to be essentially self-adjoint when the operators A-i and A+i map $\mathfrak{A}$ onto dense subspaces of $\mathfrak{H}$.

Whereas in applications the Hermitian character of an operator may be easy to establish, it is often difficult to prove that an operator is essentially self-adjoint.

If A on $\mathfrak{A}$ is any Hermitian operator, then the operators $R_i = (A-i)^{-1}$ and $R_{-i} = (A+i)^{-1}$ are defined; but their domains $\mathfrak{D}_{R_i}$ and $\mathfrak{D}_{R_{-i}}$, respectively, might be subspaces which are not dense in $\mathfrak{H}$. It is clear that the operators A-i and A+i map $\mathfrak{A}$ in a one-to-one way onto $\mathfrak{D}_{R_i}$ and $\mathfrak{D}_{R_{-i}}$ respectively, however. If A on $\mathfrak{A}$ is essentially self-adjoint, then $\mathfrak{D}_{R_i}$ and $\mathfrak{D}_{R_{-i}}$ are dense subspaces of $\mathfrak{H}$.

A Hermitian operator A on $\mathfrak{A}$ is said to be self-adjoint if and only if A-i and A+i each map $\mathfrak{A}$ onto $\mathfrak{H}$, i.e. for each element f in $\mathfrak{H}$ there exist elements u and $\tilde{u}$ in $\mathfrak{A}$ such that

$$(A-i)u = f \quad , \ (A+i)\tilde{u} = f \quad .$$

Let $\mathfrak{H}$ be the Hilbert space of functions u(x,y,z) where $\iiint_{-\infty}^{\infty} |u|^2 d\tau$ exists in the sense of Lebesgue and is finite*, with the usual inner product $(v,u) = \iiint_{-\infty}^{\infty} \bar{v} u d\tau$. We consider Schrödinger's operator

$$Hu = -(u_{xx} + u_{yy} + u_{zz}) - \frac{2Z}{r} u$$

defined on the domain

$\mathfrak{D}_H$: (1) u(x,y,z) has continuous second derivatives in the whole x,y,z-space,

(2) $\iiint_{-\infty}^{\infty} |u|^2 d\tau < \infty \quad , \quad \iiint_{-\infty}^{\infty} |\Delta u + \frac{2Z}{r} u|^2 d\tau < \infty \quad .$

It can easily be proved that H on $\mathfrak{D}_H$ is a Hermitian operator, and it is also true (though not trivial to prove) that H on $\mathfrak{D}_H$ is essentially self-adjoint.

An operator B on $\mathfrak{B}$ is said to be an extension of A on $\mathfrak{A}$ when $\mathfrak{B} \supseteq \mathfrak{A}$ and Bu = Au for every element u in $\mathfrak{A}$. It is called a trivial extension if, for each element u in $\mathfrak{B}$, there exists a sequence (u_n) of elements in $\mathfrak{A}$ such that $u_n \rightarrow u$ and $Au_n \rightarrow Bu$. A trivial extension of a Hermitian operator is Hermitian.

Every Hermitian extension of an essentially self-adjoint operator is a trivial extension.

---

* By $d\tau$ we mean dxdydz.

Proof: Suppose that A on $\mathfrak{A}$ is essentially self-adjoint, and that B on $\mathfrak{B}$ is a Hermitian extension of A on $\mathfrak{A}$. Let u be any element in $\mathfrak{B}$, and define the element f as f = Bu - iu. Since the set of elements (A-i)u with u in $\mathfrak{A}$ is dense, there exists a sequence (u_n) in $\mathfrak{A}$ such that $(A-i)u_n \longrightarrow f$. Call $(A-i)u_n = f_n$, and we have

$$f - f_n = Bu - Au_n - i(u - u_n) \quad .$$

Using the Hermitian character of B, we obtain

$$\|f - f_n\|^2 = \|B(u - u_n)\|^2 + \|u - u_n\|^2 \quad .$$

Therefore as $n \longrightarrow \infty$, we find $u_n \longrightarrow u$ and $Bu_n \longrightarrow Bu$, which proves our statement.

The domain of an essentially self-adjoint operator is not large enough in general to contain a complete set of eigenfunctions (of the point and of the continuous spectrum). The basic result is that every essentially self-adjoint operator has a Hermitian extension whose domain contains a complete set of eigenfunctions (with the continuous spectrum suitably included).

In our example we have the eigenvalue equation

$$-\Delta u - \frac{2Z}{r} u = \lambda u \quad .$$

Since the Coulomb potential $-\frac{2Z}{r}$ has a singularity at the origin, it might even be expected that a solution u of the eigenvalue equation would also be singular at the origin. We saw (see page 12) that $\phi = e^{-Zr}$ is an eigenfunction belonging to the point eigenvalue $-Z^2$. Although ϕ is continuous in the whole x,y,z-space, the derivative $\phi_x = -\frac{x}{r} e^{-Zr}$ is not continuous at the origin; and therefore ϕ is not an element of $\mathfrak{D}_H$. We define the extension H on $\mathfrak{G}$ of H on $\mathfrak{D}_H$ as follows:

$\mathfrak{C}$: (1) $u(x,y,z)$ has continuous second derivatives in $0 < x^2 + y^2 + z^2 < \infty$,

(2) $\iiint\limits_{-\infty}^{\infty} |u|^2 d\tau < \infty$, $\iiint\limits_{-\infty}^{\infty} |\Delta u + \frac{2Z}{r} u|^2 d\tau < \infty$,

(3) u, u_x, u_y, u_z are bounded in $0 < x^2 + y^2 + z^2 \leq 1$;

and

$$Hu = - \Delta u - \frac{2Z}{r} u \ .$$

(It is customary, though not strictly logical, to use the same letter H for the extension.)

It is easy to see that H on $\mathfrak{C}$ is a Hermitian extension of H on $\mathfrak{D}_H$. (Since H on $\mathfrak{D}_H$ is essentially self-adjoint, the extension is trivial.) Obviously ϕ is contained in $\mathfrak{C}$.

If A on $\mathfrak{A}$ is Hermitian, then the closure of A on $\mathfrak{A}$ is the Hermitian operator $\bar{A}$ on $\bar{\mathfrak{A}}$ which is defined as follows: $\bar{\mathfrak{A}}$ is the set of all elements u in $\mathfrak{H}$, for which there exists a sequence (u_n) in $\mathfrak{A}$ such that $u_n \longrightarrow u$ and Au_n converges. Then define $\bar{A}u = \lim\limits_{n \to \infty} Au_n$, and prove that $\bar{A}$ on $\bar{\mathfrak{A}}$ is a Hermitian operator.

The closure of a Hermitian operator is a trivial extension.

A linear operator $\bar{A}$ on $\bar{\mathfrak{A}}$ is called hypermaximal, or self-adjoint, if it is the closure of an essentially self-adjoint operator.

A Hermitian operator $\bar{A}$ on $\bar{\mathfrak{A}}$ is hypermaximal if and only if the two operators $\bar{A}-i$ and $\bar{A}+i$ map $\bar{\mathfrak{A}}$ onto the whole Hilbert space $\mathfrak{H}$.

For each hypermaximal operator A on $\bar{\mathfrak{A}}$, there is a unique resolution of the identity E_λ, $-\infty < \lambda < +\infty$, with $A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$ (v. Neumann, Stone). If $\lambda_1 < \lambda_2$ are any finite numbers and f is any element of $\mathfrak{H}$, then $(E_{\lambda_2} - E_{\lambda_1})f$ is in $\bar{\mathfrak{A}}$.

§4. Closeable operators.

A linear operator A whose domain $\mathfrak{D}_A$ is any subspace of the Hilbert space $\mathfrak{H}$ is too general a notion to be applied in functional analysis. We shall illustrate this point with a few examples. Consider the Hilbert space $\mathfrak{H}$ of complex-valued functions f(x) of the real variable x on $0 \leq x \leq 1$ for which $\int_0^1 |f(x)|^2 dx < \infty$ with the inner product $(g,f) = \int_0^1 \bar{g} f \, dx$.

Then we feel that the operator $A_1 u \equiv u''$ on

$\mathfrak{A}_1$: u, u', u'' continuous on $0 \leq x \leq 1$

is a reasonable operator, as is $A_2 u \equiv u''$ on

$\mathfrak{A}_2$: u, u' continuous and u'' piecewise continuous
in $0 \leq x \leq 1$.

Obviously A_2 is an extension—indeed a trivial one—of A_1. The equations

$$A_1 u = 0 \quad \text{with u in } \mathfrak{A}_1$$

and

$$A_2 u = 0 \quad \text{with u in } \mathfrak{A}_2$$

have the same solutions, the two linearly independent functions x and 1 which lie both in $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Now define the operator $A_3 u \equiv u''$ on

$\mathfrak{A}_3$: u continuous, u', u'' piecewise continuous on
$0 \leq x \leq 1$.

(Note that u'' is defined except possibly at a finite number when u is in $\mathfrak{A}_3$ or $\mathfrak{A}_2$, but of course u'' is assumed to be in $\mathfrak{H}$.) We immediately feel that the operator A_3 on $\mathfrak{A}_3$ is pathological. The equation

$$A_3 u = 0 \quad \text{with u in } \mathfrak{A}_3$$

has infinitely many solutions, because every continuous piecewise linear function in $0 \leq x \leq 1$ lies in $\mathfrak{A}_3$ and satisfies the above equation. We should like to exclude the operator A_3 by some requirement which does not exclude A_1 or A_2.

It is clear that it is not enough to require that the domain of the operator be dense, because $\mathfrak{A}_3$ is dense in $\mathfrak{H}$. A suitable restriction was given by v. Neumann, who introduced the important concept of a closed operator A on $\mathfrak{D}_A$.

A linear operator A defined on a dense domain $\mathfrak{D}_A$ is said to be closed when every element u in $\mathfrak{H}$ for which there is a sequence u_n in $\mathfrak{D}_A$ such that $u_n \longrightarrow u$ and Au_n converges lies in $\mathfrak{D}_A$.

It is easy to see that A_3 is not a closed operator, and consequently would be excluded if we restrict ourselves to closed operators. Unfortunately A_1 and A_2 are not closed either, but we should not like to exclude them. We can, however, replace these operators by $\bar{A}u = u''$ on

$$\bar{\mathfrak{A}}: \quad u \text{ continuous, } u' \text{ absolutely continuous, } \int_0^1 |u''|^2 dx < \infty ,$$

which is a closed extension of A_1 and A_2. But since we wish to avoid the problem of finding the closures of operators (like A_1 or A_2) which occur in applications, it is convenient to work instead with "closeable" operators.

A linear operator A defined on a dense domain $\mathfrak{D}_A$ is said to be closeable if whenever u_n is a sequence in $\mathfrak{D}_A$ and f is an element in $\mathfrak{H}$ such that $u_n \longrightarrow 0$ and $Au_n \longrightarrow f$ then it follows that $f = 0$.

It is clear from the definition that an operator which is not closeable is "very discontinuous".

Plainly an operator is closeable if it possesses a closeable extension (which may be closed). Bounded operators and Hermitian operators are closeable.

To show that A_3 on $\mathfrak{A}_3$ is not closeable, consider a sequence $s_n(x)$ of continuous, piecewise linear functions which converge uniformly to x^2 in $0 \leq x \leq 1$. Let us call $x^2 - s_n(x) = u_n$. Then $u_n \longrightarrow 0$ and $Au_n = 2 \longrightarrow 2 \neq 0$.

Moreover we shall now show that A_1 on $\mathfrak{A}_1$ and A_2 on $\mathfrak{A}_2$ are closeable. It suffices to prove that A_2 is closeable because A_2 is an extension of A_1. Suppose that u_n is a sequence in $\mathfrak{A}_2$ and f is an element in $\mathfrak{H}$ such that $u_n \longrightarrow 0$ and $A_2u_n \longrightarrow f$. Then $\lim\limits_{n \to \infty} (w, A_2u_n) = (w,f)$ for every element w in $\mathfrak{H}$. In particular this is true for all w in the subspace $\mathfrak{D}$ of functions with continuous second derivatives and such that $w(0) = w'(0) = w(1) = w'(1) = 0$. In that case

$$(w, A_2u_n) = \int_0^1 \overline{w}\, u_n'' dx = \int_0^1 \overline{w''} u_n dx \quad ;$$

and hence

$$(w,f) = \lim_{n \to \infty} (w, A_2u_n) = \lim_{n \to \infty} \int_0^1 \overline{w''} u_n dx = 0 \quad ,$$

because $u_n \longrightarrow 0$. Since $(w,f) = 0$ for all w in $\mathfrak{D}$, a dense subspace of $\mathfrak{H}$, we can conclude that $f = 0$. Therefore A_2 on $\mathfrak{A}_2$ is closeable.

If A on $\mathfrak{D}_A$ is closeable, we can define a trivial extension in the following way: Let us denote by $\overline{\mathfrak{A}}$ the set of all elements u in $\mathfrak{H}$ for which there is a sequence z_n in $\mathfrak{D}_A$ and an element g in $\mathfrak{H}$ such that

$$\lim_{n \to \infty} \|z_n - u\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|Az_n - g\| = 0 \quad .$$

If y_n is another sequence in $\mathfrak{D}_A$ and h is an element in $\mathfrak{H}$ such that $\lim_{n\to\infty} \|y_n - u\| = 0$ and $\lim_{n\to\infty} \|Az_n - h\| = 0$, then $u_n = z_n - g_n$ is a sequence in $\mathfrak{D}_A$ and $f = g - h$ is an element in $\mathfrak{H}$ such that $\lim_{n\to\infty} \|u_n\| = 0$ and $\lim_{n\to\infty} \|Au_n - f\| = 0$. Since A on $\mathfrak{D}_A$ is closeable, it follows that $f = 0$; and therefore $g = h$. Thus if u is in $\overline{\mathfrak{A}}$, but not an element of $\mathfrak{D}_A$, we can define $Au = g = h$, and this definition is independent of the particular sequence z_n or y_n selected. Evidently we obtain a linear operator A on $\overline{\mathfrak{A}}$ which is a trivial extension of A on $\mathfrak{D}_A$. Moreover it is easy to show that A on $\overline{\mathfrak{A}}$ is a closed operator. Thus we have proved:

Every closeable operator has a closed trivial extension. On the other hand, it follows immediately from the definition that:

An operator, defined on a dense domain, which has a closed extension is closeable.

Associated with a linear operator A defined on a dense domain $\mathfrak{D}_A$, there is an *adjoint* operator A^* on $\mathfrak{D}_{A^*}$ defined in the following way: *$\mathfrak{D}_{A^*}$ is the subspace of all elements v in $\mathfrak{H}$ for which there exists an element f in $\mathfrak{H}$ such that $(v, Au) = (f, u)$ for every element u in $\mathfrak{D}_A$. (Since $\mathfrak{D}_A$ is dense in $\mathfrak{H}$, there is at most one f in $\mathfrak{H}$ to each v in $\mathfrak{H}$.) For v in $\mathfrak{D}_{A^*}$ define $A^*v = f$.*

An operator with dense domain is closeable if and only if the domain of its adjoint is dense.

The proof (due to v. Neumann for "closed" rather than "closeable" operators) runs as follows:

Part I. Suppose that A on $\mathfrak{D}_A$ has an adjoint A^* on $\mathfrak{D}_{A^*}$ with $\mathfrak{D}_{A^*}$ dense in $\mathfrak{H}$. If u_n is a sequence in $\mathfrak{D}_A$ and f is an element of $\mathfrak{H}$ such that $\lim\limits_{n\to\infty} \|u_n\| = 0$, $\lim\limits_{n\to\infty} \|Au_n - f\| = 0$, then $(v,f) = \lim\limits_{n\to\infty} (v,Au_n) = \lim\limits_{n\to\infty} (Av,u_n) = 0$ for every element v in $\mathfrak{D}_{A^*}$. Since $\mathfrak{D}_{A^*}$ is dense, it follows that $f = 0$. Therefore A on $\mathfrak{D}_A$ is closeable.

Part II. Suppose that A on $\mathfrak{D}_A$ is closeable and that $\mathfrak{D}_A$ is dense in $\mathfrak{H}$. Then A^* on $\mathfrak{D}_{A^*}$ is defined. We shall denote by $\mathcal{H}$ the Hilbert space of all couples $[f,g]$ where f and g are any elements of $\mathfrak{H}$ with the inner product

$$([g,h],[g',h']) = (g,g') + (h,h') \quad .$$

Let $\mathcal{S}$ be the subspace consisting of all elements $[A^*v,-v]$ with v in $\mathfrak{D}_{A^*}$, and let $\mathcal{T}$ be the subspace consisting of all elements $[u,Au]$ with u in $\mathfrak{D}_A$. Plainly $\mathcal{S}$ is orthogonal to $\mathcal{T}$ because $(A^*v,u) - (v,Au) = 0$. Let us denote by $\bar{\mathcal{T}}$ the subspace of $\mathcal{H}$ consisting of all elements orthogonal to $\mathcal{S}$. Obviously $\mathcal{T}$ is contained in $\bar{\mathcal{T}}$, and we want to show that $\mathcal{T}$ is dense in $\bar{\mathcal{T}}$ (in the sense of the metric of $\mathcal{H}$). If this were not the case, there would exist an element $[g,h]$, not $[0,0]$, in $\bar{\mathcal{T}}$ which is orthogonal to $\mathcal{T}$, i.e. $(g,u) + (h,Au) = 0$ for all u in $\mathfrak{D}_A$. Furthermore any element in $\bar{\mathcal{T}}$ is orthogonal to $\mathcal{S}$ so that $(g,A^*v) - (h,v) = 0$ for all v in $\mathfrak{D}_{A^*}$. The first equation implies that h is in $\mathfrak{D}_{A^*}$ and that $A^*h = -g$. The second then yields for $v = h$ the result that $h = g = 0$. Since this is contrary to our assumption that $[g,h] \neq [0,0]$, we have proved that $\mathcal{T}$ is dense in $\bar{\mathcal{T}}$.

Now if $\mathcal{D}_{A^*}$ were not dense in $\mathfrak{H}$, then there would exist an element $f \neq 0$ in $\mathfrak{H}$ such that $(v,f) = 0$ for all v in $\mathcal{D}_{A^*}$. Hence $[0,f]$ is orthogonal to $\mathcal{S}$ and consequently is in $\bar{\mathcal{T}}$. Since $\mathcal{T}$ is dense in $\bar{\mathcal{T}}$, there exists a sequence u_n in $\mathcal{D}_A$ such that $[u_n, Au_n] \rightarrow [0,f]$. This means that $u_n \rightarrow 0$ and $Au_n \rightarrow f$; but since $f \neq 0$, this contradicts our assumption that A on $\mathcal{D}_A$ is closeable. Therefore $\mathcal{D}_{A^*}$ is dense in $\mathfrak{H}$.

We sum up our results. If A is a linear operator defined on a dense domain, then the following four statements are equivalent:

(1) A is closeable.

(2) A has a closed trivial extension.

(3) A has a closed extension.

(4) The domain of the adjoint A^* of A is dense.

§5. Regular (not necessarily bounded) operators.

For a family of bounded operators $A(\varepsilon)$ defined on the whole Hilbert space $\mathfrak{H}$, one can define regularity of $A(\varepsilon)$ (with respect to the perturbation parameter ε) as follows: $A(\varepsilon)$ is said to be regular in a neighborhood of $\varepsilon = 0$ when for each element u in $\mathfrak{H}$, independent of ε, the element $A(\varepsilon)u$ is regular in that neighborhood of $\varepsilon = 0$. In general if $A(\varepsilon)$ is linear but not necessarily bounded, then the domain $\mathfrak{D}(\varepsilon)$ also varies with ε. Suppose one were to define regularity as in the bounded case by saying that $A(\varepsilon)$ is regular in a neighborhood of $\varepsilon = 0$ when for each element u in $\mathfrak{D}(\varepsilon)$, independent of ε, the element $A(\varepsilon)u$ is a regular function of ε in that neighborhood of $\varepsilon = 0$. Unfortunately it might happen that the only element u in $\mathfrak{D}(\varepsilon)$, independent of ε for ε in some neighborhood of $\varepsilon = 0$, is the element zero, in which case the definition has no significant meaning.

We now define regularity in the general case.

Definition 1. A family of linear operators $A(\varepsilon)$ on the domains $\mathfrak{A}(\varepsilon)$ defined for each ε in a complex (real) neighborhood of $\varepsilon = 0$ is said to be regular with respect to ε in a complex (real) neighborhood of $\varepsilon = 0$ when the following two conditions are satisfied in a complex (real) neighborhood of $\varepsilon = 0$:

(1) *There exist bounded regular operators $U(\varepsilon)$ mapping the whole Hilbert space $\mathfrak{H}$ in a one-to-one way onto $\mathfrak{A}(\varepsilon)$.*

(2) *The element $A(\varepsilon)U(\varepsilon)$ is a regular element for each element f in $\mathfrak{H}$ which is independent of ε.*

If $\mathfrak{A}(\varepsilon) = \mathfrak{H}$ and $A(\varepsilon)$ on $\mathfrak{H}$ is bounded, we can put $U(\varepsilon) = 1$; and this is in agreement with our earlier definition (see page 73).

A sufficient condition for the regularity of an operator $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is afforded by

Criterion 1. Suppose that γ is a complex number (independent of ε) such that in a neighborhood of $\varepsilon = 0$ the operator $A(\varepsilon) + \gamma$ has a regular and bounded inverse with domain $\mathfrak{H}$. Then $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is regular in a neighborhood of $\varepsilon = 0$.

The proof is immediate. Call $R(\varepsilon) = A(\varepsilon) + \gamma)^{-1}$. Then $R(\varepsilon)$ is a bounded operator, regular in a neighborhood of $\varepsilon = 0$, which maps $\mathfrak{H}$ onto $\mathfrak{A}(\varepsilon)$. For each element f in $\mathfrak{H}$, the element $A(\varepsilon)R(\varepsilon)f = f - \gamma R(\varepsilon)f$ is regular in a neighborhood of $\varepsilon = 0$. Thus the criterion is established.

Theorem 1 below will show that Criterion 1 is not only sufficient for regularity, but nearly necessary.

We shall first prove the following

Lemma. *Suppose that A is a linear operator defined on a dense subspace $\mathfrak{A}$ of the Hilbert space $\mathfrak{H}$. Suppose that the domain $\mathfrak{A}^*$ of the adjoint operator A^* is also dense. If U is a bounded operator with domain $\mathfrak{H}$ and range $\mathfrak{A}$, then AU is a bounded operator with domain $\mathfrak{H}$.*

Remark. If A is Hermitian, then $\mathfrak{A}$ is dense from the definition; and $\mathfrak{A}^*$, which contains $\mathfrak{A}$, is also dense.

This lemma is a generalization (and consequence) of the important theorem that every linear operator A, whose domain, as well as that of its adjoint A^*, coincides with the whole Hilbert space, is bounded (Stone, page 61).

Proof. The operator $B = AU$ is defined on $\mathfrak{H}$. Let us denote by B^* on $\mathfrak{B}^*$ the adjoint of B on $\mathfrak{H}$, and by U^* on $\mathfrak{H}$ the adjoint of U on $\mathfrak{H}$. For any element v in $\mathfrak{A}^*$ and any element u in $\mathfrak{H}$, we have

$$(v,Bu) = (v,AUu) = (U^*A^*v,u) = (f,u)$$

where we have set $f = U^*A^*v$. Consequently $\mathfrak{A}^*$ is contained in $\mathfrak{B}^*$, and hence $\mathfrak{B}^*$ is dense. We shall prove in fact that $\mathfrak{B}^* = \mathfrak{H}$. Suppose that w is any element of $\mathfrak{H}$, and v_n is a sequence in $\mathfrak{B}^*$ converging to w, i.e. $\lim\limits_{n\to\infty} \|v_n - w\| = 0$. Since $(v_n,Bu) = (f_n,u)$, where $f_n = B^*v_n$, we have $\lim\limits_{n\to\infty} (f_n,u)$ exists for every element u in $\mathfrak{H}$. Thus the sequence f_n is "weakly convergent". From a well-known theorem[1], it follows that there exists an element g such that $\lim\limits_{n\to\infty} (f_n,u) = (g,u)$ for every element u in $\mathfrak{H}$. Since $(v_n,Bu) = (f_n,u)$, we obtain $(w,Bu) = (g,u)$ for all u in $\mathfrak{H}$.

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[1] Sz.-Nagy [1], pp. 9-10.

Thus w is an element of $\mathfrak{B}^*$, and therefore $\mathfrak{B}^* = \mathfrak{H}$. We have now shown that B is an operator whose domain is $\mathfrak{H}$ and the domain of B^* is also $\mathfrak{H}$. Therefore B on $\mathfrak{H}$ is bounded, and the lemma is proved.

We are now able to make clearer the connections between regularity of the operator $A(\varepsilon)$ and regularity of the inverse $(A(\varepsilon) + \gamma)^{-1}$ where γ is some suitable constant.

Theorem 1. Suppose that $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is regular in a neighborhood of $\varepsilon = 0$. If γ is a complex number (independent of ε) for which $A(0) + \gamma$ has a bounded inverse with domain $\mathfrak{H}$, then $(A(\varepsilon) + \gamma)^{-1}$ is a bounded operator with domain $\mathfrak{H}$ regular in a neighborhood of $\varepsilon = 0$.

Proof. Call $R(0) = (A(0) + \gamma)^{-1}$. Let $U(\varepsilon)$ be a bounded regular operator which maps $\mathfrak{H}$ onto $\mathfrak{A}(\varepsilon)$ and such that $A(\varepsilon)U(\varepsilon)u$ is a regular element for every element u in $\mathfrak{H}$ independent of ε. It follows from the above lemma that $V(\varepsilon) = (A(\varepsilon) + \gamma)U(\varepsilon)$ is a bounded operator on $\mathfrak{H}$, and $V(\varepsilon)$ is regular in a neighborhood of $\varepsilon = 0$. Since $V(0) = (A(0) + \gamma)U(0)$, we find $R(0)V(0) = U(0)$. If u is an element in $\mathfrak{H}$ such that $V(0)u = 0$, then $U(0)u = 0$; and consequently $u = 0$ because $U(0)$ maps $\mathfrak{H}$ in a one-to-one way onto $\mathfrak{A}(0)$. Thus $V(0)$ has an inverse, which we call $W(0)$. Suppose that f is any element in $\mathfrak{H}$. Then $R(0)f$ is in $\mathfrak{A}(0)$, and there is an element u in $\mathfrak{H}$ such that $U(0)u = R(0)f$. Hence $V(0)u = (A(0) + \gamma)U(0)u = f$; the range of $V(0)$ is $\mathfrak{H}$. Therefore since $W(0)$ is the inverse of a bounded operator defined on $\mathfrak{H}$ and the domain of $W(0)$ is

also $\mathfrak{H}$, it follows[1] that $W(0)$ is bounded. Moreover, since $V(\varepsilon)$ is a bounded operator regular in a neighborhood of $\varepsilon = 0$ and $V(0)^{-1} = W(0)$ is a bounded operator with domain $\mathfrak{H}$, it easily follows[2] that $V(\varepsilon)$ has a bounded and regular inverse $W(\varepsilon)$ in a neighborhood of $\varepsilon = 0$. Therefore $R(\varepsilon) = U(\varepsilon)W(\varepsilon)$ is a bounded operator with domain $\mathfrak{H}$ regular in a neighborhood of $\varepsilon = 0$, and $R(\varepsilon)$ is the inverse of $A(\varepsilon) + \gamma$, which is the statement that was to be proved.

If one is interested in the spectral resolution of the perturbed operator $A(\varepsilon)$ on $\mathfrak{D}(\varepsilon)$, then it is necessary to know that the operator is hypermaximal. Suppose it is known that the unperturbed operator $A(0)$ on $\mathfrak{D}(0)$ is hypermaximal. Can one conclude that $A(\varepsilon)$ on $\mathfrak{D}(\varepsilon)$ is also hypermaximal, at least in some real neighborhood of $\varepsilon = 0$? The answer is yes, and is given by

Theorem 2. Suppose that $A(\varepsilon)$ on $\mathfrak{D}(\varepsilon)$ is Hermitian for real ε and regular in a neighborhood of $\varepsilon = 0$. Suppose that $A(0)$ on $\mathfrak{D}(0)$ is hypermaximal. Then $A(\varepsilon)$ on $\mathfrak{D}(\varepsilon)$ is hypermaximal in a real neighborhood of $\varepsilon = 0$.

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[1] If f is any element in $\mathfrak{H}$, then there is an element u such that $V(0)u = f$. Thus $(W(0)f,g) = (u,g)$. If g is in the range of the adjoint V^* of $V(0)$, then there is an element $\mathfrak{H}$ such that $g = V^*h$; and we have $(W(0)f,g) = (u,V^*h) = (Vu,h) = (f,h)$. Thus the domain of the adjoint of $W(0)$ contains the range of V^*. This range is dense because if $(h,V^*w) = 0$ for all w in the range of V^* then $(Vh,w) = 0$, and consequently $Vh = 0$ and $h = 0$. Therefore the domain of $W(0)$ is dense. We can now apply our lemma with $A = W(0)$ and $U = 1$ to conclude that $W(0)$ is bounded.

[2] Cf. Rellich [3], III, page 558, Hilfsatz 4.

Proof. The operator $A(0)+\gamma$ on $\mathfrak{A}(0)$ has a bounded inverse with domain $\mathfrak{H}$ for $\gamma = i$ and $\gamma = -i$. According to Theorem 1 the operators $A(\varepsilon)+i$ and $A(\varepsilon)-i$ on $\mathfrak{A}(\varepsilon)$ have bounded inverse operators each with domain $\mathfrak{H}$ in a neighborhood of $\varepsilon = 0$. This proves the theorem.

If $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is a regular Hermitian operator and if $A(0)$ on $\mathfrak{A}(0)$ is hypermaximal then the isolated point eigenvalues of finite multiplicity of $A(0)$ and the associated eigenelements behave (when perturbed) exactly as in the case when $A(\varepsilon)$ is bounded. We state and prove

Theorem 3. *Suppose that $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is regular in a neighborhood of $\varepsilon = 0$, and Hermitian for real ε. Suppose that $A(0)$ on $\mathfrak{A}(0)$ is hypermaximal. Suppose that λ is an eigenvalue of finite multiplicity h of the operator $A(0)$, and suppose there are positive numbers d_1 and d_2 such that the spectrum of $A(0)$ in the open interval $\lambda - d_1 < \mu < \lambda + d_2$ consists exactly of the point eigenvalue λ. Then there exist power series*

$$\lambda_1(\varepsilon), \ldots, \lambda_h(\varepsilon)$$

and power series in Hilbert space

$$\phi^{(1)}(\varepsilon), \ldots, \phi^{(h)}(\varepsilon)$$

all convergent in a neighborhood of $\varepsilon = 0$, which satisfy the following conditions:

(1) *The element $\phi^{(i)}(\varepsilon)$ is an eigenelement of $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ belonging to the eigenvalue $\lambda_i(\varepsilon)$, i.e.*

$$A(\varepsilon)\phi^{(i)}(\varepsilon) = \lambda_i(\varepsilon)\phi^{(i)}(\varepsilon) \quad , \quad i = 1, \ldots, h \quad .$$

Furthermore $\lambda_i(0) = \lambda$, $i = 1, \ldots, h$; and for real ε the eigenelements form an orthonormal set, i.e. for real ε

$$(\phi^{(i)}(\varepsilon), \phi^{(j)}(\varepsilon)) = \delta_{ij}\ , \quad i, j = 1, \ldots, h \quad .$$

(2) *For each pair of positive numbers d_1', d_2' with $d_1' < d_1$ and $d_2' < d_2$, there exists a positive number ρ such that the spectrum of $A(\varepsilon)$ on $\mathfrak{M}(\varepsilon)$ in $\lambda_1 - d_1' \leq \mu \leq \lambda_1 + d_2'$ for real ε with $|\varepsilon| < \rho$ consists exactly of the points $\lambda_1(\varepsilon), \ldots, \lambda_h(\varepsilon)$.*

Remark. We admit $h = 0$; namely, we admit the possibility that the spectrum of $A(0)$ is empty in $\lambda - d_1 < \mu < \lambda + d_2$. In that case the conclusion of the theorem is that the spectrum of $A(\varepsilon)$ is empty in $\lambda - d_1' \leq \mu \leq \lambda + d_1'$ for real ε in $|\varepsilon| < \rho$.

Proof. By Theorem 2, the operator $A(\varepsilon)$ on $\mathfrak{M}(\varepsilon)$ is hypermaximal in a real neighborhood of $\varepsilon = 0$. Suppose that γ is a real number with $\lambda - d_1' < \gamma < \lambda$. Then $A(0) - \gamma$ has a bounded inverse with domain $\mathfrak{H}$, and therefore (by Theorem 1), $A(\varepsilon) - \gamma$ has a bounded regular inverse $R(\varepsilon)$ in a neighborhood of $\varepsilon = 0$, Hermitian for real ε. It can be shown that the spectrum of $R(0)$ is empty for $\mu < \frac{1}{\lambda - d_1 - \gamma}$, and for $\mu > \frac{1}{\lambda + d_2 - \gamma}$ consists only of the isolated point eigenvalue $\mu = \frac{1}{\lambda - \gamma}$, which is of multiplicity h. Therefore by Theorem 1 of §2, the spectrum for $\mu \geq \frac{1}{\lambda - d_2' - \gamma}$ consists exactly of the h eigenvalues $\pi_1(\varepsilon), \ldots, \pi_h(\varepsilon)$, which are regular analytic functions of ε for $|\varepsilon| < \rho$; with $\pi_i(0) = \frac{1}{\lambda - \gamma}$, $i = 1, \ldots, h$. There are corresponding eigenelements $\phi^{(1)}(\varepsilon), \ldots, \phi^{(h)}(\varepsilon)$ which are power series in Hilbert space for $|\varepsilon| < \rho$, and for real ε they form an orthonormal set. The spectrum of $A(\varepsilon)$ in the interval $\lambda - d_1' \leq \mu \leq \lambda + d_2'$ consists of the eigenvalues $\lambda_i(\varepsilon) = \gamma + \frac{1}{\pi_i(\varepsilon)}$ with the corresponding eigenelements $\phi^{(i)}(\varepsilon)$, which are in $\mathfrak{M}(\varepsilon)$. The $\lambda_i(\varepsilon)$ are regular in a neighborhood of $\varepsilon = 0$, and $\lambda_i(0) = \lambda$.

We now state and prove

Criterion 2. Suppose that the operator $A(\varepsilon)$ on $\mathfrak{A}$ is Hermitian in a real neighborhood of $\varepsilon = 0$ and that $\mathfrak{A}$ is independent of ε. Suppose that $A(\varepsilon)u$ is a regular element for each u in $\mathfrak{A}$. Suppose that $A(0)$ on $\mathfrak{A}$ is hypermaximal. Then $A(\varepsilon)$ on $\mathfrak{A}$ is regular in a real neighborhood of $\varepsilon = 0$.

Proof. The operator $A(0) + i$ maps $\mathfrak{A}$ onto the whole Hilbert space $\mathfrak{H}$, and $R = (A(0) + i)^{-1}$ is a bounded operator mapping $\mathfrak{H}$ onto $\mathfrak{A}$. In Definition 1 of regularity (see page 95) we can take $U(\varepsilon) = R$. Then $A(\varepsilon)U(\varepsilon)f = A(\varepsilon)Rf$ is obviously a regular element for each f in $\mathfrak{H}$, independent of ε. Therefore Criterion 2 is established.

To understand this criterion better, let us examine the meaning of the condition that "$A(\varepsilon)u$ is a regular element for each u in $\mathfrak{A}$". This means that for each u in $\mathfrak{A}$, there exist elements $v_0, v_1, v_2, \ldots$ such that

$$A(\varepsilon)u = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots$$

is a convergent power series in $\mathfrak{H}$ in some neighborhood of $\varepsilon = 0$. The elements $v_0, v_1, v_2, \ldots$ depend on u, and it follows from the uniqueness theorem about power series that there are Hermitian operators $A_0, A_1, A_2, \ldots$, defined on $\mathfrak{A}$ such that

$$A(\varepsilon)u = A_0 u + \varepsilon A_1 u + \varepsilon^2 A_2 u + \ldots \quad ,$$

for each u in $\mathfrak{A}$, is a convergent power series in some neighborhood of $\varepsilon = 0$. Plainly $A(0) = A_0$. In perturbation theory we frequently consider the case in which $A_2 = A_3 = \ldots = 0$, and $A(\varepsilon)u = A_0 u + \varepsilon A_1 u$. Then the condition that $A(\varepsilon)u$ be a regular element is trivially fulfilled.

The major condition in Criterion 2 is that the operators $A_1, A_2, \ldots$ must be defined on the domain $\mathfrak{A}$ on which $A_0 = A(0)$ is hypermaximal. Consider for example Schrödinger's operator $H(\varepsilon) = H_0 + \varepsilon H_1$,

$$H_0 u = -(u_{xx} + u_{yy} + u_{zz}) - \frac{2Z}{r} u \quad ,$$

$$H_1 u = s(x,y,z) u \quad ,$$

where s is a given continuous real-valued function. The Hilbert space $\mathfrak{H}$ is the space of complex-valued functions f(x,y,z) defined on the whole x,y,z-space for which

$$\mathfrak{H}: \qquad \iiint_{-\infty}^{\infty} |f(x,y,z)|^2 d\tau < \infty$$

with the usual inner product

$$(g,f) = \iiint_{-\infty}^{\infty} \overline{g} f \, d\tau \quad .$$

The domain $\mathfrak{D}$ of $H(\varepsilon)$ is the subspace of $\mathfrak{H}$ consisting of functions u(x,y,z) such that

$\mathfrak{D}$:
(1) u has continuous second partial derivatives,
(2) $u \equiv 0$ for $x^2 + y^2 + z^2$ sufficiently large.

The operators H_0 and H_1 are each Hermitian on $\mathfrak{D}$, and the unperturbed operator $H_0 = H(0)$ is even essentially self-adjoint on $\mathfrak{D}$. All the conditions of Criterion 2 are satisfied except that H(0) on $\mathfrak{D}$ is not hypermaximal. To try to apply Criterion 2, we can extend the essentially self-adjoint operator H_0 on $\mathfrak{D}$ to a hypermaximal operator H_0 on $\mathfrak{A} \supseteq \mathfrak{D}$. The question then arises as to whether H_1 on $\mathfrak{D}$ can be

extended to a Hermitian operator H_1 on $\mathfrak{A}$. In case H_1 on $\mathfrak{D}$ is a bounded (Hermitian) operator then it has a Hermitian extension to the whole Hilbert space $\mathfrak{H}$, and a fortiori to $\mathfrak{A}$. This is the situation if the given function $s(x,y,z)$ is bounded on the whole x,y,z-space. In this important case Criterion 2 can be applied; and Theorem 3 shows that the point eigenvalues of the perturbed operator—the point eigenvalues of $H(\varepsilon)$ are all isolated and of finite multiplicity—are power series in ε for small $|\varepsilon|$, and the same is true for the eigenfunctions.

In the Stark effect we have $s(x,y,z) = x$. In this case H_1 is defined on $\mathfrak{D}$. H_1 on $\mathfrak{D}$ has no Hermitian extension with domain $\mathfrak{A}$, however. This can be proved indirectly using Theorem 3. If there were such an extension of H_1, then there would exist point eigenvalues of the hypermaximal operator $H(\varepsilon)$ on $\mathfrak{A}$. But $H(\varepsilon)$ has no point eigenvalues (see Introduction, pp. 12-13). Therefore H_1 has no Hermitian extension with domain $\mathfrak{A}$.

§6. Examples and more criteria.

In the Hilbert space

$$\mathfrak{H}: \quad \int_0^1 |f(x)|^2 dx < \infty \quad , \quad (g,f) = \int_0^1 \bar{g}\, f\, dx \quad ,$$

consider the two Hermitian operators

$$A(\varepsilon)u = -u'' \text{ on } \mathfrak{A}(\varepsilon): \begin{array}{l} (1)\ u,\ u',\ u'' \text{ continuous in } 0 \le x \le 1 \\ (2)\ u(0) = 0,\ \varepsilon u'(1) + u(1) = 0, \end{array}$$

and

$$B(\varepsilon)u = -u'' \text{ on } \mathfrak{B}(\varepsilon): \begin{array}{l} (1)\ u,\ u',\ u'' \text{ continuous in } 0 \le x \le 1. \\ (2)\ u(0) = 0,\ u'(1) + \varepsilon u(1) = 0 \ . \end{array}$$

We ask whether or not these operators are regular near $\varepsilon = 0$. Now, for continuous functions f in $0 \leq x \leq 1$, it is easy to verify that

$$A^{-1}(\varepsilon)f = \int_0^1 G(x,y;\varepsilon)f(y)dy \ , \quad B^{-1}(\varepsilon)f = \int_0^1 K(x,y;\varepsilon)f(y)dy \ ,$$

where

$$G(x,y;\varepsilon) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| - \frac{1}{1+\varepsilon}xy \quad ,$$

$$K(x,y;\varepsilon) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| - \frac{\varepsilon}{1+\varepsilon}xy \quad .$$

Let us denote by $\mathfrak{C}$ the subspace of all functions $f(x)$ continuous in $0 \leq x \leq 1$. We note that $A(\varepsilon)\,\mathfrak{A}(\varepsilon) = \mathfrak{C}$ and $B(\varepsilon)\,\mathfrak{B}(\varepsilon) = \mathfrak{C}$, and the inverses $R(\varepsilon) = A^{-1}(\varepsilon)$ and $S(\varepsilon) = B^{-1}(\varepsilon)$ are bounded operators on $\mathfrak{C}$. Consequently $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ and $B(\varepsilon)$ on $\mathfrak{B}(\varepsilon)$ are essentially self-adjoint. (At this point we have used the result that a Hermitian operator A on $\mathfrak{A}$ is essentially self-adjoint if there exists a real number c such that the operator $A-c$ on $\mathfrak{A}$ has a bounded inverse with dense domain. We leave the proof of this fact as an exercise for the reader.) Obviously the operators $R(\varepsilon)$ and $S(\varepsilon)$ can be extended to $\mathfrak{H}$, and these extension are regular. Let us call $\overline{A(\varepsilon)}$ on $\overline{\mathfrak{A}(\varepsilon)}$ and $\overline{B(\varepsilon)}$ on $\overline{\mathfrak{B}(\varepsilon)}$ the corresponding extensions of $A(\varepsilon)$ and $B(\varepsilon)$. According to Definition 1, both operators $\overline{A(\varepsilon)}$ and $\overline{B(\varepsilon)}$ are regular. We would like to say that $A(\varepsilon)$ and $B(\varepsilon)$ are themselves regular.

To handle these examples, it is convenient to extend slightly the concept of regular Hermitian operators. We give another

Definition 2. Suppose that $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is a family of essentially self-adjoint operators in a (real) neighborhood of $\varepsilon = 0$. It is said to be regular if the hypermaximal extension $\overline{A(\varepsilon)}$ on $\overline{\mathfrak{A}}(\varepsilon)$ is regular in the sense of Definition 1.

To avoid some of the difficulty which occurred in applying Criterion 2, we shall establish

Criterion 3. Suppose that $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is a family of Hermitian operators in a real neighborhood of $\varepsilon = 0$, with domain $\mathfrak{A}$ independent of ε. Suppose that $A_0 = A(0)$ on $\mathfrak{A}$ is essentially self-adjoint, and there exist Hermitian operators $A_1, A_2, \ldots$ on $\mathfrak{A}$ such that for each element u in $\mathfrak{A}$

$$A(\varepsilon)u = A_0 u + \varepsilon A_1 u + \varepsilon^2 A_2 u + \ldots$$

and

$$\|A_\nu u\| \leq p^{\nu-1}(a\|u\| + b\|A_0 u\|) \ , \ \nu = 1,2,\ldots,$$

where p, a, b are three non-negative numbers. Then $A(\varepsilon)$ on $\mathfrak{A}$ is essentially self-adjoint and regular (in the sense of Definition 2) in a real neighborhood of $\varepsilon = 0$.

Remark. The inequalities in this criterion can be replaced by

$$(A_\nu u, A_\nu u) \leq p^{\nu-1}(a(u,u) + b|(u,A_0 u)|), \ \nu = 1,2,\ldots \ .$$

These require more; they imply the former inequalities, with perhaps different constants.

Proof. Let $\overline{A}_0$ on $\overline{\mathfrak{A}}$ be the hypermaximal extension of A_0 on $\mathfrak{A}$. Then we can construct trivial extensions $\overline{A}_\nu$ on $\overline{\mathfrak{A}}$ of A_ν on $\mathfrak{A}$. Since A_0 on $\mathfrak{A}$ is essentially self-adjoint, for each element u in $\overline{\mathfrak{A}}$ there exists a sequence u_n of elements in $\mathfrak{A}$ such that

$$\lim_{n\to\infty} \|u_n - u\| = 0 \quad \text{and} \quad \lim_{n,m\to\infty} \|A_0 u_n - A_0 u_m\| = 0 \quad .$$

Using the inequalities

$$\|A_\nu(u_n - u_m)\| \leq p^{\nu-1}(a\|u_n - u_m\| + b\|A_0(u_n - u_m)\|) \quad .$$

We can conclude that $\lim_{n,m\to\infty} \|A_\nu u_n - A_\nu u_m\| = 0$; and hence there exists an element f in $\mathfrak{H}$ such that $\lim_{n\to\infty} A_\nu u_n = f$. If v_n is another sequence in $\mathfrak{A}$ such that

$$\lim_{n\to\infty} \|v_n - u\| = 0 \quad \text{and} \quad \lim_{n,m\to\infty} \|A_0 v_n - A_0 v_m\| = 0 \quad ,$$

then, because

$$\|A_\nu(u_n - v_n)\| \leq p^{\nu-1}(a\|u_n - v_n\| + b\|A_0(u_n - v_n)\|) \quad ,$$

it follows that $\lim_{n\to\infty} A_\nu v_n = f$. Therefore it makes sense to define $\overline{A}_\nu u = f$. Thus we have defined an extension $\overline{A}_\nu$ on $\overline{\mathfrak{A}}$ of A_ν on $\mathfrak{A}$. The extension is trivial because to each u in $\overline{\mathfrak{A}}$ there exists a sequence u_n in $\mathfrak{A}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{n\to\infty} \|A_\nu u_n - \overline{A}_\nu u\| = 0$.

These extensions $\overline{A}_\nu$ on $\overline{\mathfrak{A}}$ again satisfy the inequalities

$$\|A_\nu u\| \leq p^{\nu-1}(a\|u\| + b\|\overline{A}_0 u\|)$$

for all u in $\overline{\mathfrak{A}}$. Hence $\overline{A(\varepsilon)}u = \overline{A}_0 u + \varepsilon\overline{A}_1 u + \varepsilon^2\overline{A}_2 u + \ldots$ is a regular element in a neighborhood of $\varepsilon = 0$. It then follows from Criterion 2 that $\overline{A(\varepsilon)}$ on $\overline{\mathfrak{A}}$ is regular near $\varepsilon = 0$; and hence by Definition 2, $A(\varepsilon)$ on $\mathfrak{A}$ is regular. This completes the proof of the criterion. The proof of the remark is trivial.

Suppose the hypotheses of Criterion 3 are satisfied, and suppose that A(0) has an isolated point eigenvalue of finite multiplicity. Then by Criterion 3 and Theorem 3 (§5, page 105) the perturbed operator $A(\varepsilon)$ possesses a point eigenvalue and a corresponding eigenelement

$$\lambda(\varepsilon) = \lambda + \varepsilon\lambda_1 + \dots \quad , \quad \phi(\varepsilon) = \phi^{(0)} + \varepsilon\phi^{(1)} + \dots$$

which are convergent power series in ε. Even if $\phi^{(0)}$ should be an element of the domain $\mathfrak{A}$ (and it may happen that $\phi^{(0)}$ lies not in $\mathfrak{A}$ but in $\bar{\mathfrak{A}}$, the domain where A(0) is hypermaximal), the higher order approximations $\phi^{(1)}, \phi^{(2)}, \dots$ may be in $\bar{\mathfrak{A}}$, not in $\mathfrak{A}$. From the relation

$$(\bar{A}_0 + \varepsilon\bar{A}_1 + \dots)(\phi^{(0)} + \varepsilon\phi^{(1)} + \dots)$$
$$= (\lambda + \varepsilon\lambda_1 + \dots)(\phi^{(0)} + \varepsilon\phi^{(1)} + \dots)$$

we obtain the equations

$$A_0\phi^{(0)} = \lambda\phi^{(0)} \quad ,$$
$$\bar{A}_0\phi^{(1)} - \lambda\phi^{(1)} = \lambda_1\phi^{(0)} - \bar{A}_1\phi^{(0)} \quad ,$$
$$\dots\dots\dots\dots\dots\dots$$

In particular, we obtain from the second equation the result that

$$\lambda_1 = (\phi^{(0)}, \bar{A}_1\phi^{(0)}) \quad .$$

As we stated previously this result may be expressed as a rule: The first order approximation to the eigenvalue is the mean value of the "first order" term $\bar{A}_1$ of the perturbed operator taken for the zero-approximation to the eigenelement.

As an example we consider the Hamiltonian of the helium atom $H_0 + \varepsilon H_1$; here H_0 and H_1 are the operators

$$H_0 u = -\tfrac{1}{2}(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3})$$

$$- \tfrac{1}{2}(u_{y_1y_1} + u_{y_2y_2} + u_{y_3y_3}) - \frac{Z}{r_x} u - \frac{Z}{r_y} u \quad ,$$

$$H_1 u = \frac{1}{r_{xy}} u \quad ,$$

where

$$r_x = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \; , \quad r_y = (y_1^2 + y_2^2 + y_3^2)^{\frac{1}{2}} \quad ,$$

$$r_{xy} = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2)^{\frac{1}{2}} \quad ;$$

the operators are defined on the subspace

$\dot{\mathfrak{D}}$: (1) $u(x_1, x_2, x_3, y_1, y_2, y_3)$ has continuous second partial derivatives,

(2) $u \equiv 0$ for sufficiently large $r_x^2 + r_y^2$,

in the Hilbert space

$$\mathfrak{H} : \int |f(x_1, x_2, x_3, y_1, y_2, y_3)|^2 d\tau_6 < \infty, \; (g, f) = \int \bar{g} f \, d\tau_6 \quad .$$

(We intend the integration to be extended over the whole space, and we mean by $d\tau_6$ the element of volume.)

It is clear that H_0 and H_1 on $\dot{\mathfrak{D}}$ are Hermitian operators. Moreover it can be proved that H_0 on $\dot{\mathfrak{D}}$ is essentially self-adjoint (see Rellich [8], Part 2, page 39). The spectrum of H_0 on $\dot{\mathfrak{D}}$ (or rather that of its hypermaximal extension) has been determined, and it is known that its lower part is

discrete. The physically important problem is to determine the point eigenvalues of $H_o + \varepsilon H_1$ for $\varepsilon = 1$, and especially to locate the lower eigenvalues. The lowest eigenvalue of the problem for

$$-\frac{1}{2}\Delta_x u - \frac{Z}{r_x} u = \lambda_x u$$

in the x_1, x_2, x_3-space is $\lambda_x = -\frac{1}{2}Z^2$ and the corresponding normalized eigenfunction is $\phi_x^{(o)} = \frac{1}{\pi^{\frac{1}{2}}} Z^{\frac{3}{2}} e^{-Zr_x}$ (Introduction, see page 12). Therefore we expect that the lowest eigenvalue of H_o is $\lambda = -Z^2$ with the corresponding eigenfunction $\phi^{(o)} = \frac{1}{\pi} Z^3 e^{-Z(r_x + x_y)}$. The lowest eigenvalue of the unperturbed operator is isolated and of finite multiplicity (in fact simple). When we prove that $H_o + \varepsilon H_1$ is regular near $\varepsilon = 0$, then application of the perturbation calculus will be justified. We would then know that

$$\lambda(\varepsilon) = \lambda + \varepsilon\lambda_1 + \dots$$

where $\lambda = -Z^2$ and

$$\lambda_1 = (\phi^{(o)}, H_1\phi^{(o)}) = \frac{Z^3}{\pi}\int \frac{1}{r_{xy}} e^{-Z(r_x + r_y)} d\tau_6 = \frac{5}{8}Z \quad ,$$

i.e.
$$\lambda(\varepsilon) = -Z^2 + \frac{5}{8}Z\varepsilon + \dots \quad .$$

Observe that this is far from what we want to know. We are interested in $\varepsilon = 1$; we would need to investigate whether or not the power series $\lambda(\varepsilon)$ converges for $\varepsilon = 1$, and of course we want to know how good the approximation $-Z^2 + \frac{5}{8}Z\varepsilon$ is.

We shall now prove that $H_o + \varepsilon H_1$ on $\dot{\mathcal{D}}$ is regular for small ε. According to Criterion 3, it suffices to prove that for some numbers a and b, the inequality

$$(H_1u,H_1u) \leq a(u,u) + b|(u,H_0u)|$$

holds for every element u in $\dot{\mathfrak{D}}$.

Every function $f(x,y,z)$ with continuous first derivatives in the whole x,y,z-space, and vanishing identically for large $r^2 = x^2 + y^2 + z^2$ satisfies the inequality

$$\int \frac{|f|^2}{r^2} d\tau_3 \leq 4\int (|f_x|^2 + |f_y|^2 + |f_z|^2) d\tau_3$$

(see Courant-Hilbert I, page 446). Consequently we obtain for $u = u(x_1,x_2,x_3,y_1,y_2,y_3)$ in $\dot{\mathfrak{D}}$ the inequalities

$$\int \frac{|u|^2}{r_x^2} d\tau_6 \leq 4\int |\mathrm{grad}_x u|^2 d\tau_6 , \quad \int \frac{|u|^2}{r_y^2} d\tau_6 \leq 4\int |\mathrm{grad}_y u|^2 d\tau_6 .$$

If we put $x = y_1 - x_1$, $y = y_2 - x_2$, $z = y_3 - x_3$, and $r = r_{12}$ and $u(x_1,x_2,x_3,y_1,y_2,y_3) = u(x_1,x_2,x_3,x_1+x,x_2+y,x_3+z) = f(x,y,z)$, and apply the inequality, we find

$$\int \frac{|u(x_1,x_2,x_3,x_1+x,x_2+y,x_3+z)|^2}{x^2+y^2+z^2} dxdydz$$

$$\leq 4\int (|u_{y_1}|^2 + |u_{y_2}|^2 + |u_{y_3}|^2) dxdydz$$

or

$$\int \frac{|u(x_1,x_2,x_3,y_1,y_2,y_3)|^2}{r_{12}^2} dy_1dy_2dy_3 \leq 4\int |\mathrm{grad}_y u|^2 dy_1dy_2dy_3 .$$

Finally we obtain

$$\int \frac{|u|^2}{r_{12}^2} d\tau_6 \leq 4\int |\mathrm{grad}_y u|^2 d\tau_6 .$$

Of course we can derive in exactly the same way the inequality

$$\int \frac{|u|^2}{r_{12}^2}\, d\tau_6 \leq 4 \int |\mathrm{grad}_x u|^2 d\tau_6 \quad .$$

Hence we can write

$$\int \frac{|u|^2}{r_{12}^2}\, d\tau_6 \leq 2\left(\int |\mathrm{grad}_x u|^2 d\tau_6 + \int |\mathrm{grad}_y u|^2 d\tau_6\right) \quad .$$

Now we have

$$(u,H_o u) = \frac{1}{2}\int |\mathrm{grad}_x u|^2 + |\mathrm{grad}_y u|^2) d\tau_6$$

$$- Z\int \frac{|u|^2}{r_x} d\tau_6 - Z\int \frac{|u|^2}{r_y} d\tau_6 \quad ,$$

and hence

$$\int (|\mathrm{grad}_x u|^2 + |\mathrm{grad}_y u|^2) d\tau_6 \leq 2(u,H_o u) + 2Z\left(\int |u|^2 d\tau_6\right)^{\frac{1}{2}}$$

$$\cdot\left[\left(\int \frac{|u|^2}{r_x^2} d\tau_6\right)^{\frac{1}{2}} + \left(\int \frac{|u|^2}{r_y^2} d\tau_6\right)^{\frac{1}{2}}\right]$$

$$\leq 2(u,H_o) + 4Z\left(\int |u|^2 d\tau_6\right)\left[\left(\int |\mathrm{grad}_x u|^2 d\tau_6\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(\int |\mathrm{grad}_y u|^2 d\tau_6\right)^{\frac{1}{2}}\right] \quad .$$

Let us use the abbreviations

$$\alpha = 2Z\left(\int |u|^2 d\tau_6\right)^{\frac{1}{2}} \quad , \quad \beta = \left(\int |\mathrm{grad}_x u|^2 d\tau_6\right)^{\frac{1}{2}} \quad ,$$

$$\gamma = \left(\int |\mathrm{grad}_y u|^2 d\tau_6\right)^{\frac{1}{2}} \quad ;$$

then we have

$$\beta^2 + \gamma^2 \leq 2(u,H_0u) + 2\alpha(\beta + \gamma) \quad .$$

Since $2\alpha b \leq 2\alpha^2 + \frac{1}{2}\beta^2$ and $2\alpha\gamma \leq 2\alpha^2 + \frac{1}{2}\gamma^2$, we obtain

$$\beta^2 + \gamma^2 \leq 4(u,H_0u) + 8\alpha^2 \quad ;$$

and hence

$$\int |\text{grad}_x u|^2 d\tau_6 + \int |\text{grad}_y u|^2 d\tau_6 \leq 32Z^2 \int |u|^2 d\tau_6 + 4(u,H_0u) \quad .$$

Therefore we finally find

$$(H_1u,H_1u) \leq 64Z^2(u,u) + 8|(u,H_0u)| \quad ,$$

which proves that $H_0 + \varepsilon H_1$ on $\dot{\mathcal{D}}$ is regular near $\varepsilon = 0$.

We consider another example. We shall investigate the dependence of the eigenvalues of $\Delta u + \lambda u = 0$ on the region.[1]

Suppose that D and D^* are two closed bounded domains in n-dimensional Euclidean space, and suppose that

$$x_i^* = x_i + \varepsilon S_i(x_1,\ldots,x_n) \quad , \qquad i = 1,\ldots,n$$

is a one-to-one mapping of the fixed domain D onto the domain D^*, which depends on ε. We assume that the boundaries of D and D^* and the functions S_i are smooth enough to permit the following calculations:

We consider the eigenvalue equation

$$\Delta^* u^* + \lambda^* u^* = 0 \quad \text{in} \quad D^*$$

with $u^* = 0$ at the boundary of D^*. We shall investigate how the eigenvalues depend on ε. For each function $u^*(x_1^*,\ldots,x_n^*)$ defined in D^*, we define

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[1] See P. R. Garabedian and M. Schiffer, Convexity of domain functionals, Chapter V, Journal d'analyse mathematique, vol. 2 (1952-53), pp. 283-368. In part we use the notation of this paper.

$$u(x_1,\dots,x_n) = u^*(x_1+\varepsilon S_1,\dots,x_n+\varepsilon S_n) \quad .$$

If we denote by ξ, η, σ the matrices

$$\xi = \left(\frac{\partial x_i^*}{\partial x_k}\right) \quad , \quad \eta = \left(\frac{\partial x_i}{\partial x_k^*}\right) \quad , \quad \sigma = \left(\frac{\partial S_i}{\partial x_k}\right) , \quad i,k = 1,\dots,n \quad ,$$

then we have

$$\xi\eta = 1 \quad , \quad \xi = 1+\varepsilon\sigma \quad .$$

In addition we introduce the matrix

$$\beta = (\beta_{ik}) = \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial x_j^*}\,\frac{\partial x_k}{\partial x_j^*}\right) \quad ;$$

we note that

$$\beta = \eta\eta' = (1+\varepsilon\sigma)^{-1}(1+\varepsilon\sigma')^{-1} = 1-\varepsilon(\sigma+\sigma')+\dots \quad .$$

The determinant

$$\Theta = \det\,(\xi)$$

is equal to

$$\Theta = \det\,(1+\varepsilon\sigma) = 1+\varepsilon\left(\sum_{i=1}^{n}\frac{\partial S_i}{\partial x_i}\right)+\dots \quad ,$$

and $\Theta \neq 0$ in D.

In the following it is essential that the functions β_{ik} and Θ be the power series in ε, although the functions $x_i = x_i(x_1^*,\dots,x_n^*;\varepsilon)$ might not be regular analytic functions of ε.

Now we have

$$\frac{\int_{D^*} u^*(-\Lambda^* u^*)\,d\tau^*}{\int_{D^*} (u^*)^2\,d\tau^*} = \frac{\int_{D^*} \sum_{j=1}^{n} \left(\frac{\partial u^*}{\partial x_j^*}\right)^2 d\tau^*}{\int_{D^*} (u^*)^2\,d\tau^*}$$

$$= \frac{\int_D \sum_{i,j,k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial x_k}{\partial x_j^*} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial x_j^*} \Theta\, d\tau}{\int_D u^2 \Theta\, d\tau} .$$

The equation $-\Lambda^* u^* = \lambda^* u^*$ is transformed into

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(\Theta \beta_{ki} \frac{\partial u}{\partial x_i}\right) = \lambda^* \Theta u \quad \text{in } D ,$$

and the boundary condition $u^* = 0$ at the boundary of D^* becomes $u = 0$ at the boundary of D. In the Hilbert space

$$\mathfrak{H}: \quad \int_D |f|^2 d\tau < \infty , \qquad (g,f) = \int_D \bar{g} f\, d\tau ,$$

we consider the subspace

$\mathfrak{M}$: (1) u has continuous second derivatives in D,
(2) $u = 0$ at the boundary of D;

and we define the operators M and N on $\mathfrak{M}$ as

$$Mu = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(\Theta \beta_{ki} \frac{\partial u}{\partial x_i}\right) , \quad Nu = \Theta u .$$

Thus our eigenvalue problem assumes the form

$$Mu = \lambda^* Nu , \quad \text{with } u \text{ in } \mathfrak{M} .$$

To reduce this to the usual form $Au = \lambda^* u$, we may proceed in several ways. We can introduce a new inner product $(g,f)_N = (g,Nf)$ defined for elements g and f in $\mathfrak{M}$. If N is a positive definite operator ($(u,Nu) > 0$ for all $u \neq 0$ in $\mathfrak{M}$), then the inner product $(g,f)_N$ can be used to define in a natural way a new Hilbert space $\mathfrak{H}_N$. Moreover, if N^{-1} is a bounded operator, then $Au = \lambda^* u$ with $A = N^{-1}M$, which is Hermitian on $\mathfrak{M}$ with respect to the new inner product. A second possibility occurs when M on $\mathfrak{M}$ is positive definite and has a bounded inverse. Then we can use the inner product $(g,f)_M = (g,Mf)$ to define in a natural way a new Hilbert space $\mathfrak{H}_M$. The operator $A = M^{-1}N$ is Hermitian on $\mathfrak{M}$ with respect to the new inner product, and the eigenvalue equation becomes $Au = \lambda u$, with $\lambda = (\lambda^*)^{-1}$. In our case both procedures are possible, but the both have a certain difficulty. We have

$$\mathfrak{H}_N: \quad \int_D |u|^2 \textcircled{\mu}\, d\tau < \infty \quad , \qquad (u,v)_N = \int_D \bar{g} f \textcircled{\mu}\, d\tau \quad ,$$

$$\mathfrak{H}_M: \quad \int_D \left(\sum_{j,k=1}^{n} \beta_{kj} \frac{\partial u}{\partial x_j} \frac{\overline{\partial u}}{\partial x_k} \right) d\tau < \infty \quad ,$$

$$(u,v)_M = \int_D \left(\sum_{j,k=1}^{n} \beta_{kj} \frac{\partial u}{\partial x_j} \frac{\overline{\partial v}}{\partial x_k} \right) d\tau \quad .$$

In both cases the inner product depends on the perturbation parameter ε. The same is generally true for problems like $M(\varepsilon)u = \lambda N(\varepsilon)u$.

To avoid this difficulty, we proceed by an alternate method. If N on $\mathfrak{M}$ is a bounded positive definite operator whose inverse is bounded, then it seems possible to define Hermitian operators $N^{\frac{1}{2}}$ and $N^{-\frac{1}{2}}$ on $\mathfrak{M}$; and we may even expect that $N^{\frac{1}{2}}\mathfrak{M} = \mathfrak{M}$, i.e. to each element u in $\mathfrak{M}$, there is one

and only one element ϕ in $\mathfrak{A}$ such that $N^{\frac{1}{2}}u = \phi$. Then we have $A\phi = \lambda^{*}\phi$ with $A = N^{-\frac{1}{2}}MN^{\frac{1}{2}}$, which is Hermitian under the original ε-independent inner product (u,v).

In our case we find

$$A\phi \equiv -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(\beta_{ki}\frac{\partial \phi}{\partial x_i}\right) + q\phi$$

where

$$q \equiv \Theta^{-\frac{1}{2}} \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(\beta_{ki}\frac{\partial \Theta^{\frac{1}{2}}}{\partial x_i}\right) \quad ,$$

and the eigenvalue equation assumes the form

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(\beta_{ki}\frac{\partial \phi}{\partial x_i}\right) + q\phi = \lambda^{*}\phi$$

where ϕ is an element of the subspace $\mathfrak{A}$ defined above (page 114). Thus the operator $A = A(\varepsilon)$ on $\mathfrak{A}$ depends on ε, whereas $\mathfrak{A}$ is independent of ε. The unperturbed operator $A(0)$ on $\mathfrak{A}$ is essentially self-adjoint, and its spectrum consists entirely of isolated point eigenvalues of finite multiplicity. Therefore to each unperturbed eigenvalue λ of multiplicity h there exist[1] h perturbed eigenvalues

$$\lambda_i^{*} = \lambda + \varepsilon\lambda_{1i} + \varepsilon^2\lambda_{2i} + \ldots \quad , \qquad\qquad i = 1,\ldots,h \quad ,$$

and corresponding eigenfunctions

$$\phi_i^{*} = \phi_i^{(o)} + \varepsilon\phi_i^{(1)} + \varepsilon^2\phi_i^{(2)} + \ldots \quad , \qquad\qquad i = 1,\ldots,h \quad ,$$

provided $A(\varepsilon)$ on $\mathfrak{A}$ is regular (in the sense of Definition 2). The regularity can be proved by demonstrating that the inverse

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[1] For the case of a simple eigenvalue of the unperturbed operator, a proof is given by Garabedian and Schiffer, ibid.

operator $(A(\varepsilon))^{-1}$ is an integral operator whose kernel, Green's function, is a power series in ε, giving rise to a bounded regular operator. If one wishes to avoid theorems concerning Green's function, one can try to apply Criterion 3 (page 105 which involves estimating $\|A_\nu\phi\|$. In our case it is easier to estimate $|(\phi,A_\nu\phi)|$, and we shall establish the inequalities

$$|(u,A_\nu u)| \leq p^{\nu-1}(a(u,u) + b|(u,A_0 u)|), \quad \nu = 1,2,\ldots,$$

for some non-negative constants p, a, b. Later on we shall obtain a criterion for regularity in which these are the key inequalities, but now we merely prove them for the problem at hand.

Plainly we have

$$\beta_{ik} = \delta_{ik} + \sum_{\nu=1}^{\infty} \varepsilon^\nu \beta_{ik,\nu} \quad , \quad i,k = 1,\ldots,n \quad ,$$

and

$$q = \sum_{\nu=1}^{\infty} \varepsilon^\nu q_\nu$$

with

$$|\beta_{ik,\nu}| \leq \rho^\nu \quad , \quad |q_\nu| \leq \rho^\nu \quad , \quad \nu = 1,2,\ldots,$$

for some positive number ρ and for all $x_1,\ldots,x_n$ in D. Furthermore, we have

$$A(\varepsilon)\phi = A\phi + \sum_{\nu=1}^{\infty} \varepsilon^\nu A_\nu \phi$$

where

$$A_\nu\phi = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(\beta_{ik,\nu} \frac{\partial \phi}{\partial x_i} \right) + q_\nu \phi \quad .$$

For ϕ in $\mathfrak{A}$ we find

$$(\phi,A_\nu\phi) = \int_D \left(\left(\sum_{i,k=1}^{n} \beta_{ik,\nu} \frac{\partial \phi}{\partial x_i} \frac{\overline{\partial \phi}}{\partial x_k} \right) + q_\nu |\phi|^2 \right) d\tau \quad ,$$

and hence

$$(\phi, A_\nu \phi) \leq p^\nu \int_D \left(\sum_{i=1}^{n} \left| \frac{\partial \phi}{\partial x_i} \right|^2 + |\phi|^2 \right) d\tau$$

for some positive number p. Therefore we have proved that

$$(\phi, A_\nu \phi) \leq p^{\nu-1}(p(\phi,\phi) + p|(\phi, A(0)\phi)|), \quad \nu = 1, \ldots,$$

which can be used to establish the regularity of $A(\varepsilon)$.

We recall that

$$\beta_{ik,1} = -\left(\frac{\partial S_i}{\partial x_k} + \frac{\partial S_k}{\partial x_i} \right) \quad ,$$

and note that

$$\Theta^{\frac{1}{2}} = 1 + \varepsilon \, \frac{1}{2} \left(\sum_{j=1}^{n} \frac{\partial S_j}{\partial x_j} \right) + \ldots$$

then

$$\frac{\partial \Theta^{\frac{1}{2}}}{\partial x_i} = \varepsilon \, \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 S_j}{\partial x_i \partial x_j} + \ldots$$

so that

$$q_1 = \frac{1}{2} \Delta \left(\sum_{j=1}^{n} \frac{\partial S_j}{\partial x_j} \right) \quad .$$

Therefore we have

$$A(\varepsilon)\phi = -\Delta\phi + \varepsilon \left[\sum_{1,k=1}^{n} \frac{\partial}{\partial x_k} \left(\left(\frac{\partial S_i}{\partial x_k} + \frac{\partial S_k}{\partial x_i} \right) \frac{\partial \phi}{\partial x_i} \right) + \frac{\phi}{2} \Delta \left(\sum_{j=1}^{n} \frac{\partial S_j}{\partial x_j} \right) \right] + \ldots \quad .$$

If λ is an unperturbed eigenvalue of multiplicity h, then there exist h perturbed eigenvalues

$$\lambda^* = \lambda + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots$$

and corresponding eigenfunctions

$$\phi^* = \phi^{(0)} + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \dots \quad .$$

Then

$$\lambda_1 = (\phi^{(0)}, A_1\phi^{(0)}) = -\int_D \sum_{i,k=1}^{n} \left(\frac{\partial S_i}{\partial x_k} + \frac{\partial S_k}{\partial x_i}\right) \frac{\partial \phi^{(0)}}{\partial x_i} \frac{\overline{\partial \phi^{(0)}}}{\partial x_k}\, d\tau$$

$$+ \int_D \frac{1}{2} |\phi^{(0)}|^2 \Delta \left(\sum_{j=1}^{n} \frac{\partial S_j}{\partial x_j}\right) d\tau \quad .$$

After some transformations we obtain

$$\lambda_1 = - \int_\Gamma \left|\frac{\partial \phi^{(0)}}{\partial \nu}\right|^2 \left(\sum_{k=1}^{n} S_k \frac{\partial x_k}{\partial \nu}\right) d\Omega$$

where $\frac{\partial}{\partial \nu}$ means differentiation in the direction of the outward normal, and the integral is extended over the boundary Γ of the domain D. Thus the first approximation to the perturbed eigenvalue can be expressed in terms depending only on the values of the unperturbed eigenfunction on the boundary of D.

It should also be mentioned that the exact perturbed eigenvalue can be expressed[1] as

$$\lambda^*(\varepsilon) = \frac{1}{4} \int_{\Gamma^*} \left|\frac{\partial u^*}{\partial \nu^*}\right|^2 \frac{\partial}{\partial \nu^*} (x_1^{*2} + \dots + x_n^{*2})\, d\Omega^*$$

where u^* is a solution of the differential equation $-\Delta^* u^* = \lambda^* u^*$ in D^*, and u^* vanishes on Γ^*, and $\int_{D^*} |u^*|^2 d\tau = 1$.

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[1] F. Rellich, Darstellung du Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, Math. Zeitschrift, vol. 46 (1940), page 635. Similar formulas for $\Delta\Delta u - \lambda u = 0$ are given in Rellich [8]. page 131.

We conclude this section with the following example. Consider the Hilbert space

$$\mathfrak{H}: \quad \iiint_{-\infty}^{\infty} |f(x,y,z)|^2 d\tau < \infty , \quad (g,f) = \iiint_{-\infty}^{\infty} \bar{g} f \, d\tau \quad ,$$

and the subspace

$\mathfrak{A}$: (1) u has continuous second derivatives in the whole x,y,z-space,

(2) $u \equiv 0$ for sufficiently large $r^2 = x^2 + y^2 + z^2$.

Suppose that $s(x,y,z)$ is real-valued, continuous, and bounded in the whole x,y,z-space with $|s(x,y,z)| \leq \sigma$. Consider the operator

$$A(\varepsilon) = A_o + \varepsilon A_1$$

on $\mathfrak{A}$ with $A_o u = -\Delta u$ and $A_1 u = \frac{s(x,y,z)}{r} u$. We note that

$$\iiint_{-\infty}^{\infty} \left|\frac{s(x,y,z)}{r} u\right|^2 d\tau \leq \sigma^2 \iiint_{-\infty}^{\infty} \frac{|u|^2}{r^2} d\tau \leq 4\sigma^2 \iiint_{-\infty}^{\infty} |\text{grad } u|^2 d\tau \quad ,$$

and hence

$$(A_1 u, A_1 u) \leq 4\sigma^2 |(u, A_o u)|$$

for u in $\mathfrak{A}$. Therefore it follows from Criterion 3 that the operator $A(\varepsilon)$ on $\mathfrak{A}$ is regular.

Suppose we consider the operator $-\Delta + \varepsilon \frac{s(x,y,z)}{r^2}$. Can this operator be regular? Certainly not on $\mathfrak{A}$, because the operator is not even defined on $\mathfrak{A}$. However we could consider the operator on $\dot{\mathfrak{D}}$ the subspace of $\mathfrak{A}$ consisting of all elements in $\mathfrak{A}$ which vanish in a neighborhood of $r = 0$. Then $B_1 u = \frac{s(x,y,z)}{r^2} u$ on $\dot{\mathfrak{D}}$ is Hermitian. We ask whether

$$B(\varepsilon)u = -\Delta u + \varepsilon B_1 u$$

is regular on $\mathfrak{D}$? To apply Criterion 3 we would have to prove that for every u in $\mathfrak{D}$,

$$\|B_1 u\| \leq G\|u\| + b\|\Delta u\|$$

for some constants a and b. As a matter of fact in the next section we shall prove such an inequality with a = 0 and determine the best constant $c = b^{-1}$ for which

$$c \iiint_{-\infty}^{\infty} \frac{|u|^2}{r^4}\, d\tau \leq \iiint_{-\infty}^{\infty} |\Delta u|^2 d\tau$$

for u in $\mathfrak{D}$. Nonetheless Criterion 3 does not apply because $B(0) = -\Delta$ is not essentially self-adjoint in $\mathfrak{D}$.

§7. The inequality $c\int r^{-4}|u|^2 d\tau \leq \int |\Delta u|^2 d\tau$.

We shall establish this inequality in the domain $\mathfrak{D}$ of all complex-valued functions $u = u(x_1, \ldots, x_m)$ which have continuous derivatives of all orders and which vanish identically outside $a \leq r \leq b$, where a and b are some positive numbers and $r = (x_1^2 + \ldots + x_m^2)^{\frac{1}{2}}$. The numbers a and b can vary with the function u. The problem is to find the best constant c for which the inequality holds for all u in $\mathfrak{D}$.

This problem is connected with several kinds of questions concerning Hermitian operators. We define the Hilbert space

$$\mathfrak{H}: \int_{0<r<\infty} r^{-4}|f(x_1, \ldots, x_m)|^2 d\tau < \infty, \qquad (g,f) = \int_{0<r<\infty} r^{-4}\bar{g} f\, d\tau$$

and consider the eigenvalue equation

$$\Delta\Delta u = \lambda r^{-4} u \quad \text{for } u \text{ in } \mathfrak{D}.$$

The operator $Bu = r^4 \Delta\Delta u$ is Hermitian on the subspace $\mathfrak{D}$ of $\mathfrak{H}$, and finding the best constant c for the inequality amounts to finding the lowest point of the spectrum of the operator B on $\mathfrak{D}$ or of a suitable hypermaximal extension.

Alternatively we can consider the Hilbert space

$$\mathfrak{H}: \quad \int_{0<r<\infty} |f(x_1,\dots,x_m)|^2 d\tau < \infty , \quad (g,f) = \int_{0<r<\infty} \bar{g} f d\tau$$

and the eigenvalue equation

$$\Delta\Delta u - \frac{c}{r^4} u = \lambda u \quad \text{for } u \text{ in } \mathfrak{D} .$$

The operator $Bu = \Delta\Delta u - \frac{c}{r^4} u$ is Hermitian on the subspace $\mathfrak{D}$ of the Hilbert space $\mathfrak{H}$. Now we ask for the largest number c for which the operator B on $\mathfrak{D}$ is bounded below. This means that we seek the largest constant c for which there exists some constant a such that

$$c \int_{0<r<\infty} r^{-4}|u|^2 d\tau \leq \int_{0<r<\infty} |\Delta u|^2 d\tau + a \int_{0<r<\infty} |u|^2 d\tau$$

for all u in $\mathfrak{D}$. (This is not quite the original inequality, but it is closely connected to it.) Of course this inequality is intimately related to the perturbation problem

$$-\Delta u + \frac{\varepsilon}{r^2} u = \lambda u$$

and the existence of constants a and b such that

$$\int_{0<r<\infty} r^{-4}|u|^2 d\tau \leq a \int_{0<r<\infty} |u|^2 d\tau + b \int_{0<r<\infty} |\Delta u|^2 d\tau .$$

We proceed to

Theorem 1. Suppose that $u = u(x_1,\dots,x_m)$ is a function in $\mathfrak{D}$ which is not identically zero, and in case $m = 2$ suppose that

$$\int_0^{2\pi} u(r\cos\phi, r\sin\phi)\cos\phi\, d\phi = 0 \;,$$

$$\int_0^{2\pi} u(r\cos\phi, r\sin\phi)\sin\phi\, d\phi = 0 \;, \qquad 0 < r < \infty \;.$$

Then

$$\frac{(m-4)^2 m^2}{16} \int_{0<r<\infty} r^{-4}|u|^2 d\tau < \int_{0<r<\infty} |\Delta u|^2 d\tau \;.$$

For each set of three constants c, a, R such that

$$R > 0 \;, \quad c > 0 \text{ if } m = 2, \; c > \frac{(m-4)^2 m^2}{16} \text{ if } m = 1,3,4,\ldots,$$

there exist a function u_0 in $\mathfrak{D}$ which vanishes identically for $r > R$ and a function u_∞ in $\mathfrak{D}$ which vanishes identically for $r < R$ satisfying the inequalities

$$c \int_{0<r<\infty} r^{-4}|u_0|^2 d\tau > \int_{0<r<\infty} |\Delta u_0|^2 d\tau + a \int_{0<r<\infty} |u_0|^2 d\tau \;,$$

$$c \int_{0<r<\infty} r^{-4}|u_\infty|^2 d\tau > \int_{0<r<\infty} |\Delta u_\infty|^2 d\tau \;.$$

Heuristics.

It plainly suffices to prove the theorem for real functions in $\mathfrak{D}$, and in the following all functions are assumed to be real.

We not only wish to prove the theorem, but we want to indicate a method for finding the best constant c, which can be applied in similar problems.

One might expect that the best constant c will appear even when the functions u in $\mathfrak{D}$ are restricted to the subset $\mathfrak{D}_r$ of functions $u = f(r)$ which depend only on r. (This turns out to be wrong in case $m = 2$, however.) For the function $u = f(r)$ the equation

$$\Delta\Delta u - \frac{c}{r^4} u = \lambda u$$

reduces to

$$\left(-\frac{d^2}{dr^2} + \frac{m-1}{r}\frac{d}{dr}\right)\left(-\frac{d^2}{dr^2} + \frac{m-1}{r}\frac{d}{dr}\right) f - \frac{c}{r^4} f = \lambda f \quad ,$$

which is a fourth order ordinary differential equation with a regular singular point at $r = 0$. Now one knows* that a second order ordinary differential operator with a regular singular point at $r = 0$ ceases to be bounded below at the point where the characteristic exponents (which describe the singularities of the solutions near $r = 0$) are no longer real. In the case of a fourth order ordinary differential equation some such relation between the realness of the characteristic exponents and the nature of the spectrum has not yet been established, but is conjectured. The characteristic exponents are found by trying a solution of the form

$$f = r^{\alpha} + a_1 r^{\alpha+1} + a_2 r^{\alpha+2} + \dots \; .$$

Inserting this series into our equation, we find (from the fact that the coefficient of $r^{\alpha-4}$ vanishes)

$$\alpha(\alpha+m-2)(\alpha-2)(\alpha+m-4) - c = 0 \quad .$$

Regarding $c = c(\alpha)$ as a function of α, we can sketch the graph of $c(\alpha)$ for a typical value of m.

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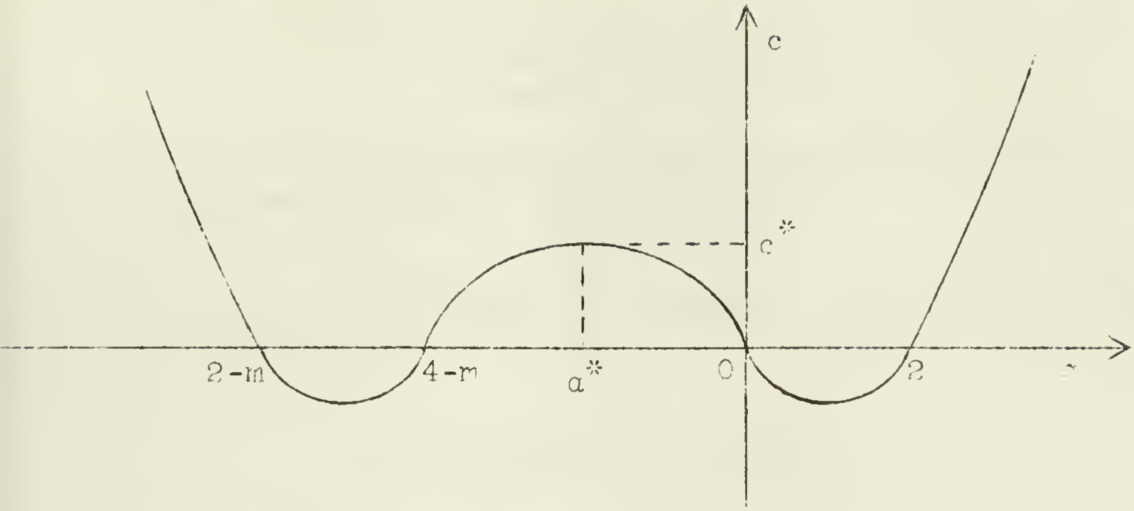
* F. Rellich, "Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung", Math. Ann., vol. 122 (1951) pp. 343-368. Hilfssatz 2, p. 381.



graph of c(α)

Since c(α) is symmetric with respect to the line $\alpha = \alpha^* = \frac{4-m}{2}$ and c(α) tends to $+\infty$ as $\alpha \to \pm\infty$, it follows that $c^* = c(\alpha^*) = \frac{m^2(m-4)^2}{16}$ is a relative maximum and is the largest value of c for which all the characteristic exponents are real. Thus we expect $c = \frac{m^2(m-4)^2}{16}$ to be the best constant for our inequality. Note that for this value of c and $\lambda = 0$, the function $u = r^{\frac{4-m}{2}}$ is a solution of our ordinary differential equation. We now begin the

Proof of Theorem 1. We shall use spherical harmonics Y_k of dimension n; in particular we use the fact that they are a complete set of functions.

Suppose that u is a function of Ω. Consider Δu on a fixed sphere $r = \rho$. For a complete orthonormal system of real spherical harmonics $Y_o, Y_1, Y_2, \ldots$, we have Parseval's relation

$$\int_{r=\rho} (\Delta u)^2 d\omega = \sum_{k=0}^{\infty} c_k^2(\rho) \quad ,$$

where

$$c_k(\rho) = \int_{r=\rho} Y_k \Delta u \, d\omega \quad .$$

Now we can write

$$\Delta u = \left(\frac{\partial^2}{\partial r^2} + \frac{m-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda\right)u \quad ,$$

where Λ is a differential operator independent of r involving only angular variables. Hence we have

$$c_k(\rho) = \int_{r=\rho} Y_k\left(\frac{\partial^2}{\partial r^2} + \frac{m-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda\right)u\,d\omega = \left(\frac{d^2}{d\rho^2} + \frac{m-1}{\rho}\frac{d}{d\rho} - \frac{k(k+m-2)}{\rho^2}\right)f_k$$

where

$$f_k = f_k(\rho) = \int_{r=\rho} Y_k\,u\,d\omega \quad ;$$

and we obtain

$$\int_{0<r<\infty} (\Delta u)^2 d\tau = \sum_{k=0}^{\infty} \int_0^{\infty} r^{m-1} c_k^2(r)\,dr$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} [r^{m-1} f_k''^2 + (m-1)^2 r^{m-3} f_k'^2 + k^2(k+m-2)^2 r^{m-5} f_k^2$$

$$+ 2(m-1)r^{m-2} f_k'' f_k' - 2k(k+m-2) r^{m-3} f_k'' f_k$$

$$- 2(m-1)k(k+m-2) r^{m-4} f_k' f_k]\,dr$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} (r^{m-1} f_k''^2 + a r^{m-3} f_k'^2 + b r^{m-5} f_k^2)\,dr \quad ,$$

where

$$a = 2k(k+m-2) + m-1 \quad \text{and} \quad b = k(k+m-2)[k(k+m-2) + 2(m-4)] \quad .$$

Suppose that f(r) is any function which vanishes identically near $r = 0$ and $r = \infty$ and which has continuous second derivatives in $0 < r < \infty$. Note that each $f_k(r)$, $k = 1, 2, \ldots,$ has these properties. We shall find another representation for

$$\int_0^{\infty} (r^{m-1} f''^2 + a r^{m-3} f'^2)\,dr \quad .$$

We use the function $r^{\frac{4-m}{2}}$ which occurred in our heuristic argument to define a new function $g(r)$ by the relation

$$f(r) = r^{\frac{4-m}{2}} g(r) \quad .$$

This artifice was known to Jacobi. We find that

$$f' = \frac{4-m}{2} r^{\frac{2-m}{2}} g + r^{\frac{4-m}{2}} g'$$

and

$$f'' = \frac{(4-m)(2-m)}{4} r^{-\frac{m}{2}} g + (4-m) r^{\frac{2-m}{2}} g' + r^{\frac{4-m}{2}} g'' \quad .$$

Thus we obtain

$$\int_0^\infty (r^{m-1} f''^2 + a r^{m-3} f'^2) dr$$

$$= \int_0^\infty [r^{-1} g^2 ((2 - \tfrac{m}{2})^2 (1 - \tfrac{m}{2})^2 + (2 - \tfrac{m}{2})^2 a) + r g'^2 ((4-m)^2 + a)$$

$$+ r^3 g''^2 + g g' ((4-m)^2 (1 - \tfrac{m}{2}) + (4-m) a)$$

$$+ r g g'' (4-m)(1 - \tfrac{m}{2}) + 2 r^2 g' g'' (4-m)] dr$$

$$= \frac{(4-m)^2 m^2}{16} \int_0^\infty r^{-1} g^2 dr + \int_0^\infty [r^{-1} g^2 \frac{(4-m)^2}{4} (1-m+a)$$

$$+ r g'^2 (a + \tfrac{1}{2}(m-4)(m-2)) + r^3 g''^2] dr \quad .$$

Now put $g_k(r) = r^{\frac{4-m}{2}} f_k$; then we have

$$\int_0^\infty (r^{m-1} f_k''^2 + a r^{m-3} f_k'^2 + b r^{m-5} f_k^2) dr$$

$$= \frac{(m-4)^2 m^2}{16} \int_0^\infty g_k^2 r^{-1} dr + \int_0^\infty (r^3 g_k''^2 + p_1 r g_k'^2 + p_o r^{-1} g_k^2) dr$$

where

$$p_o = \frac{(m-4)^2}{4}(a-m+1) + b = k(k+m-2)\left[\frac{(m-4)^2}{2} + k(k+m-2) + 2(m-4)\right]$$

$$p_1 = 2k(k+m-2) + m-1 + \tfrac{1}{2}(m-4)(m-2) \quad .$$

In case m = 1, we have $p_o = k(k-1)[k(k-1) - \frac{3}{2}]$ and $p_1 = 2k(k-1) + \frac{3}{2}$. Plainly $p_o \geq 0$, $p_1 \geq 0$ for all $k = 0,1,2,\dots$.
In case m = 2, we have $p_o = k^2(k^2-2)$ and $p_1 = 2k^2+1$. Plainly $p_o \geq 0$, $p_1 \geq 0$ for $k = 0,2,3,\dots$, but $p_o < 0$ for $k = 1$.
In case m = 3, we have $p_o = k(k+1)[k(k+1) - \frac{3}{2}]$ and $p_1 = 2k(k+1) + \frac{3}{2}$. Plainly $p_o \geq 0$, $p_1 \geq 0$ for all $k = 0,1,2,\dots$.
In case m = 4, we plainly have $p_o \geq 0$, $p_1 \geq 0$ for all $k = 0,1,2,\dots$.

Hence we have

$$\int_0^\infty (r^{m-1}f_k''^2 + ar^{m-3}f_k'^2 + br^{m-5}f_k^2)\,dr$$
$$\geq \frac{(m-4)^2m^2}{16} \int_0^\infty g_k^2 r^{-1}\,dr + \int_0^\infty r^3 g_k''^2\,dr$$

for $k = 0,1,2,\dots$ provided $m \neq 2$ and for $k = 0,2,3,\dots$ if $m = 2$. If $\int_0^\infty r^3 g_k''^2\,dr = 0$ then $g_k'' \equiv 0$; and, since g_k is identically zero near $r = 0$ and $r = \infty$, it even follows that $g_k \equiv 0$. Thus if $\int_0^\infty r^3 g_k''^2\,dr = 0$ for $k = 0,1,2,\dots$, then $u \equiv 0$. Hence we have

$$\int_0^\infty (r^{m-1}f_k''^2 + ar^{m-3}f_k'^2 + br^{m-5}f_k^2)\,dr > \frac{(m-4)^2m^2}{16} \int_0^\infty r^{m-5}f_k\,dr$$

if $u \not\equiv 0$ and if $f_1(r) \equiv 0$ in case $m = 2$. We finally obtain

$$\int_{0<r<\infty} (\Delta u)^2 d\tau > \frac{(m-4)^2m^2}{16} \int_{0<r<\infty} r^{-4}u^2 d\tau$$

for real $u \not\equiv 0$ and, in case $m = 2$, $\int\limits_{r=\rho} Y_1 u\,d\omega = 0$ for $0 < \rho < \infty$ and all spherical harmonics Y_1 of order 1 and dimension 2, i.e. $Y_1 = \cos\phi$ and $Y_1 = \sin\phi$ where $x = r\cos\phi$ and $y = r\sin\phi$. This proves the first part of the theorem.

To prove the rest of the theorem, we first introduce functions which depend only on $r = (x_1^2 + \ldots + x_m^2)^{\frac{1}{2}}$, which vanish identically near $r = 0$ and $r = \infty$, which are identically equal to 1 except near $r = 0$ and $r = \infty$, and which have continuous partial derivatives of all orders with respect to $x_1, \ldots, x_m$. We define the auxiliary function

$$e(x) = \begin{cases} 0 & , \text{ for } x < -1 , \\ e_0 \int\limits_{-1}^{x} \exp\left(\frac{1}{x^2-1}\right)dx & , \text{ for } -1 \leq x \leq 1 , \\ 1 & , \text{ for } x > 1 , \end{cases}$$

where the constant e_0 is chosen so that $e(1) = 1$, i.e. $e_0 = \left(\int\limits_{-1}^{1} \exp\left(\frac{1}{x^2-1}\right)dx\right)^{-1}$. It is easy to see that $e(x)$ has continuous derivatives of all orders in $-\infty < x < \infty$, and $e(x)$ satisfies the inequalities

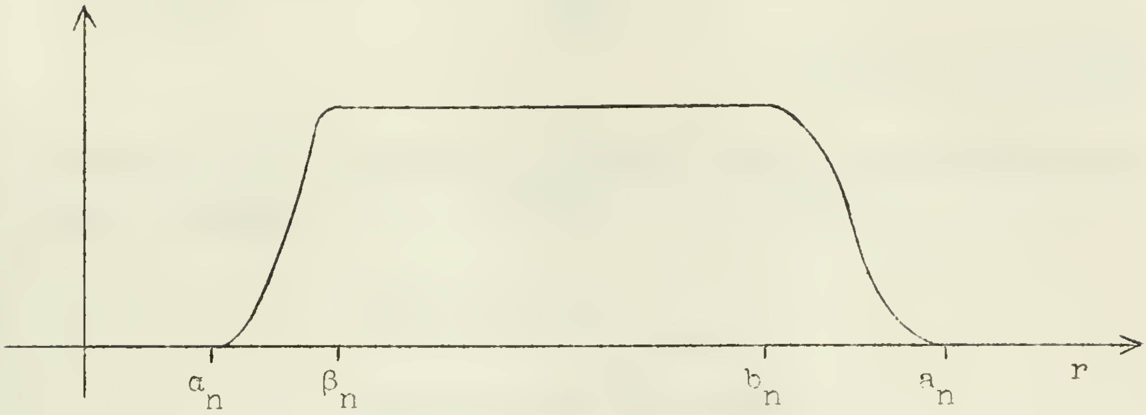
$$0 \leq e(x) \leq 1 , \quad 0 \leq e'(x) \leq e_0 , \quad |e''(x)| \leq 4e_0 ,$$

for $-\infty < x < \infty$.

Suppose that $0 < \alpha_n < \beta_n < b_n < a_n$. Define

$$h_n(r) = e\left(\frac{2r - \alpha_n - \beta_n}{\beta_n - \alpha_n}\right) e\left(\frac{2r - a_n - b_n}{b_n - a_n}\right) ,$$

and note that $h_n(r)$ has continuous derivatives of all orders in $0 < r < \infty$.



Graph of $h_n(r)$, $0 < r < \infty$.

Evidently $h_n(r) = 0$ in $0 < r < \alpha_n$ and in $a_n < r < \infty$, and $h_n(r) = 1$ in $\beta_n < r < b_n$, and $0 \leq h_n(r) \leq 1$ in $0 < r < \infty$. Moreover

$$|h_n'(r)| \leq \frac{2e_o}{\beta_n - \alpha_n} , \quad |h_n''(r)| \leq \frac{16e_o}{(\beta_n - \alpha_n)^2} , \quad \text{for } \alpha_n \leq r \leq \beta_n ,$$

$$|h_n'(r)| \leq \frac{2e_o}{a_n - b_n} , \quad |h_n''(r)| \leq \frac{16e_o}{(a_n - b_n)^2} , \quad \text{for } b_n \leq r \leq a_n .$$

Now suppose that R and c are given numbers with $R > 0$ and $c > \frac{(m-4)^2 m^2}{16}$. In order to construct the function u_o occurring in the statement of Theorem 1, we put $b_n = \frac{R}{2}$, $a_n = R$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{1}{n}$, for $n \geq n_o > \frac{4}{R}$, and we define $v_n(x_1, \ldots, x_m) = r^{2-\frac{m}{2}} h_n(r)$. In $\beta_n < r < \infty$, the function v_n is independent of n, and there we denote v_n by v.

For $r > 0$ we have

$$\Delta v_n = (2 - \tfrac{m}{2}) \tfrac{m}{2} r^{-\frac{m}{2}} h_n + 3r^{1-\frac{m}{2}} h_n' + r^{2-\frac{m}{2}} h_n'' .$$

Then we find

$$\int_{0<r<\frac{1}{n}} |\Delta v_n|^2 d\tau \leq 3\omega_m \int_{\frac{1}{2n}<r<\frac{1}{n}} r^{-1}[(2-\tfrac{m}{2})^2 \frac{m^2}{4} h_n^2 + 9(rh_n')^2 + (r^2h_n'')^2]dr$$

where ω_m is the volume of the unit sphere in m-dimensional space. In $\frac{1}{2n} < r < \frac{1}{n}$, we have the estimates

$$|h_n(r)| \leq 1 \quad ,$$

$$|rh_n'(r)| \leq \frac{1}{n} \frac{2e_o}{\frac{1}{2n}} = 4e_o \quad ,$$

$$|r^2h_n''(r)| \leq \frac{1}{n^2} \frac{16e_o}{(\frac{1}{2n})^2} = 64e_o \quad .$$

Hence it follows that there is a constant C_1 independent of n such that

$$\int_{0<r<\frac{1}{n}} |\Delta v_n|^2 d\tau \leq C_1 \quad .$$

(In the following constants independent of n will be denoted by C_1, C_2, C_3, etc.) Furthermore we have:

$$\int_{\frac{1}{n}<r<\frac{R}{2}} |\Delta v_n|^2 d\tau = \omega_m (2-\tfrac{m}{2})^2 \frac{m^2}{4} \int_{\frac{1}{n}}^{\frac{R}{2}} r^{-1}dr = \omega_m (2-\tfrac{m}{2})^2 \frac{m^2}{4} \log \frac{Rn}{2}$$

$$\int_{0<r<\frac{1}{n}} r^{-4}|v_n|^2 d\tau \leq \omega_m \int_{\frac{1}{2n}}^{\frac{1}{n}} r^{-1}dr \leq C_2 \quad ,$$

$$\int_{\frac{1}{n}<r<\frac{R}{2}} r^{-4}|v_n|^2 d\tau = \omega_m \int_{\frac{1}{n}}^{\frac{R}{2}} r^{-1}d\tau = \omega_m \log \frac{Rn}{2}$$

$$\int_{\frac{R}{2}<r<\infty} r^{-4}|v_n|^2 d\tau = \omega_m \int_{\frac{R}{2}}^{R} r^{-1} h_n^2(r)\,dr \leq C_3 \quad ,$$

$$\int_{0<r<\infty} |v_n|^2 d\tau \leq \omega_m \int_0^R r^3 dr \leq C_4 \quad .$$

Therefore, for each real number a, we find

$$\lim_{n\to\infty} \frac{\int_{0<r<\infty} |\Delta v_n|^2 d\tau + a \int_{0<r<\infty} |v_n|^2 d\tau}{\int_{0<r<\infty} r^{-4}|v_n|^2 d\tau} = (2-\tfrac{m}{2})^2 \, \frac{m^2}{4} \quad .$$

If n is chosen sufficiently large, then v_n serves as the function u_o whose existence is asserted in Theorem 1.

In case m = 2 we can prove a little more. Put $v_n(x_1,x_2) = x_1 h_n(r) = (r \cos \phi) h_n(r)$; here r, ϕ are polar coordinates in the x_1, x_2-plane. We obtain

$$\Delta v_n = (r h_n'' + 3h_n')\cos \phi$$

so that

$$\int_{0<r<\infty} |\Delta v_n|^2 d\tau = \pi \int_0^R r^{-1}(r^2 h_n'' + 3r h_n')^2 dr$$

$$\leq 2\pi[(16e_o)^2 + (12e_o)^2]\left(\int_{\frac{1}{2n}}^{\frac{1}{n}} + \int_{\frac{R}{2}}^{R}\right) r^{-1} dr \leq C_5 \quad .$$

Furthermore we have

$$\int_{0<r<\infty} |v_n|^2 d\tau \leq \pi \int_0^R r^3 dr \leq C_6$$

and

$$\int\limits_{0<r<\infty} r^{-4}|v_n|^2 d\tau \geq \int\limits_{\frac{1}{n}<r<\frac{R}{2}} r^{-4}|v_n|^2 d\tau = \pi \int_{\frac{1}{n}}^{\frac{R}{2}} r^{-1} dr = \pi \log \frac{Rn}{2} \quad .$$

Therefore for each real number a, we find

$$\lim_{n\to\infty} \frac{\int\limits_{0<r<\infty} |\Delta v_n|^2 d\tau + a \int\limits_{0<r<\infty} |v_n|^2 d\tau}{\int\limits_{0<r<\infty} r^{-4}|v_n|^2 d\tau} = 0 \quad .$$

To construct the function u_∞, we choose $\alpha_n = R$, $\beta_n = 2R$, $b_n = n$, $a_n = 2n$, with $n \geq n_0 > 2R$, and we again define $v_n(x_1,\ldots,x_m) = r^{2-\frac{m}{2}} h_n(r)$. We find

$$\int\limits_{n<r<\infty} |\Delta v_n|^2 d\tau \leq 3\omega_m \int_n^{2n} r^{-1}[(2-\tfrac{m}{2})^2 \tfrac{m^2}{4} h_n^2 + 9(rh_n')^2 + (r^2h_n'')^2]dr.$$

The inequalities $|h_n(r)| \leq 1$, $|rh_n'(r)| \leq 2n \frac{2e_0}{n} = 4e_0$, and $|r^2h_n''(r)| \leq 4n^2 \frac{16e_0}{n^2} = 64e_0$ hold in $n < r < 2n$, and hence

$$\int\limits_{n<r<\infty} |\Delta v_n|^2 d\tau \leq C_7 \quad .$$

It is easy to see that

$$\int\limits_{0<r<2R} |\Delta v_n|^2 d\tau \leq C_8 , \qquad \int\limits_{2R<r<n} |\Delta v_n|^2 d\tau = \omega_m (2-\tfrac{2}{m})^2 \frac{m^2}{4} \int_{2R}^{n} r^{-1} dr,$$

$$\int\limits_{0<r<2R} r^{-4}|v_n|^2 d\tau \leq C_9 , \qquad \int\limits_{2R<r<n} r^{-4}|v_n|^2 d\tau = \omega_m \int_{2R}^{n} r^{-1} dr ,$$

$$\int\limits_{n<r<\infty} r^{-4}|v_n|^2 d\tau \leq \omega_m \int_n^{2n} r^{-1} dr \leq C_{10} \quad .$$

Hence we find

$$\lim_{n\to\infty} \frac{\int\limits_{0<r<\infty} |\Delta v_n|^2 d\tau}{\int\limits_{0<r<\infty} r^{-4}|v_n|^2 d\tau} = \left(2-\frac{m}{2}\right)^2 \frac{m^2}{4} \quad .$$

For some sufficiently large n, the function v_n serves as u_∞ in Theorem 1.

This completes the proof of Theorem 1.

§8. Hermitian forms.

Let $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$ be two subspaces of a Hilbert space $\mathfrak{H}$. Then a (bilinear) form A(v,u) on $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ is a functional which assigns a complex number to each ordered pair u, v, with u in $\mathfrak{F}^{(1)}$ and v in $\mathfrak{F}^{(2)}$ in such a way that

$$A(v,a_1u_1+a_2u_2) = a_1A(v,u_1)+a_2A(v,u_2), \quad u,\ u_1,\ u_2 \text{ in } \mathfrak{F}^{(1)}$$

$$A(a_1v_1+a_2v_2,u) = \bar{a}_1A(v_1,u)+\bar{a}_2A(v_2,u), \quad v,\ v_1,\ v_2 \text{ in } \mathfrak{F}^{(2)}.$$

The form A(v,u) is called Hermitian if $\mathfrak{F}^{(1)} = \mathfrak{F}^{(2)} = \mathfrak{F}_A$ is a dense subspace and if $A(v,u) = \overline{A(u,v)}$ for all u, v in $\mathfrak{F}_A$. The subspace $\mathfrak{F}_A$ is called the domain of the form A(v,u).

If A on $\mathfrak{D}_A$ is a linear operator, we can define $A(v,u) = (v,Au)$ for v on $\mathfrak{H}$, u on $\mathfrak{D}_A$; and thus we have a bilinear form in $\mathfrak{F}^{(1)} = \mathfrak{H}$, $\mathfrak{F}^{(2)} = \mathfrak{D}$. If A on $\mathfrak{D}_A$ is a Hermitian operator we obtain the Hermitian form $A(v,u) = (v,Au)$ on $\mathfrak{F}_A = \mathfrak{D}_A$.

At the beginning of section 6 we considered the operators

$$A(\varepsilon)u = -u'' \text{ on } \mathfrak{D}(\varepsilon)\text{: (1) } u,\ u',\ u'' \text{ continuous in } 0 \le x \le 1\ ,$$
$$\text{(2) } u(0) = 0,\ \varepsilon u'(1)+u(1) = 0,$$

and

$$B(\varepsilon)u = -u'' \text{ on } \mathfrak{B}(\varepsilon)\text{: } \begin{array}{l} (1)\ u,\ u',\ u'' \text{ continuous in } 0 \le x \le 1 \\ (2)\ u(0) = 0,\ u'(1) + \varepsilon u(1) = 0, \end{array}$$

in the Hilbert space

$$\mathfrak{H}: \int_0^1 |f|^2 dx < \infty \quad , \quad (g,f) = \int_0^1 \bar{g} f \, dx \quad .$$

Obviously

$$(v,Au) = \int_0^1 \bar{v}' u' dx + \frac{1}{\varepsilon} \overline{v(1)} u(1) \ , \quad u,\ v \text{ in } \mathfrak{A}(\varepsilon)$$

$$(v,Bu) = \int_0^1 \bar{v}' u' dx + \varepsilon \overline{v(1)} u(1) \ , \quad u,\ v \text{ in } \mathfrak{B}(\varepsilon)$$

are Hermitian forms for real $\varepsilon \neq 0$. They can be extended to Hermitian forms with the same domain

$$\mathfrak{L}: \begin{array}{l} (1)\ u,\ u',\ u'' \text{ continuous in } 0 \le x \le 1 \\ (2)\ u(0) = 0 \end{array}$$

where

$$\left.\begin{array}{l} A(v,u) = \int_0^1 \bar{v}' u' dx + \frac{1}{\varepsilon} \overline{v(1)} u(1) \ , \\ B(v,u) = \int_0^1 \bar{v}' u' dx + \varepsilon \overline{v(1)} u(1) \ . \end{array}\right\} \quad u,\ v \text{ in } \mathfrak{L}$$

In the definition of $\mathfrak{A}(\varepsilon)$ there are two conditions of different character. The first condition "u, u', u" continuous in $0 \le x \le 1$" is somewhat vague and can be replaced by different conditions e.g. "u, u' continuous, u" piecewise continuous" without changing the operator essentially. However, the second condition cannot be altered without changing the operator essentially (i.e. without changing its

spectrum). The representation by means of the form separates these two conditions (as far as the right end of the interval is concerned). It is easier to compare the two forms $A(v,u)$ on $\mathcal{L}'$ and $B(v,u)$ on $\mathcal{L}$ than to compare the two operators $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ and $B(\varepsilon)$ on $\mathfrak{B}(\varepsilon)$.

The concept of the extension of a form can be defined generally as follows: *The Hermitian form $B(v,u)$ on $\mathcal{L}_B$ is said to be an extension of the Hermitian form $A(v,u)$ on $\mathcal{L}_A$ when $\mathcal{L}_B \supseteqq \mathcal{L}_A$ and $B(v,u) = A(v,u)$ for u, v in $\mathcal{L}_A$. It is called a trivial extension if to each pair v, u in $\mathfrak{B}$ there exist a pair of sequences u_n in $\mathfrak{A}$ with*

$$\lim_{n\to\infty} \|u_n - u\| = 0 , \qquad \lim_{n\to\infty} \|v_n - v\| = 0 ,$$

$$\lim_{m,n\to\infty} A(u_n-u_m, u_n-u_m) = 0 , \qquad \lim_{n,m\to\infty} A(v_n-v_m, v_n-v_m) = 0 ,$$

and such that

$$B(v,u) = \lim_{n\to\infty} A(v_n, u_n) \quad .$$

Exercise: Prove that the form $A(v,u) = \int_0^1 \bar{v}'u'dx + \frac{1}{\varepsilon}\bar{v}(1)u(1)$ on $\mathcal{L}$ is a trivial extension of the form $(v, A(\varepsilon)u)$ on $\mathfrak{A}(\varepsilon)$.

In quantum mechanics Dirac's δ-function is sometimes used to make clear the physical significance of a boundary condition.

Consider the Hilbert space

$$\mathfrak{H}: \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty , \qquad (g,f) = \int_{-\infty}^{\infty} \overline{g(x)} f(x)dx$$

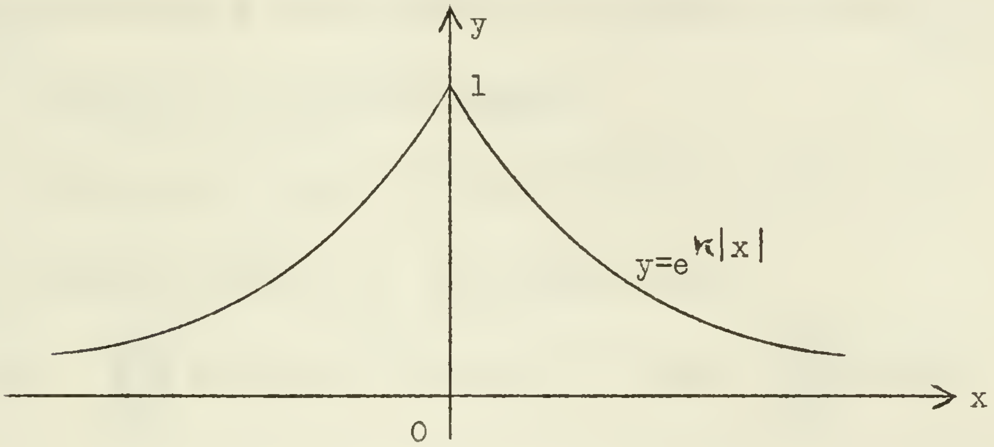
and the operator

$Au = -u''$ on $\mathfrak{A}$: (1) u continuous in $-\infty < x < \infty$, u', u'' piecewise continuous in $-\infty < x < \infty$, and continuous for $x \neq 0$;

$$\int_{-\infty}^{\infty} |u|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |u''|^2 dx < \infty .$$

$$(2) \quad -u'(+0) + u'(-0) + \kappa u(0) = 0$$

where κ is a given real number. Obviously A on $\mathfrak{A}$ has one and only one point eigenvalue if $\kappa < 0$, namely $\lambda_0 = -\kappa^2$, with the un-normalized eigenfunction $u_0 = e^{\kappa|x|}$.



If we want to interpret the operator A on $\mathfrak{A}$ as the quantum mechanical Hamiltonian of the motion of a particle along the x-axis, the significance of condition 2 in the definition of $\mathfrak{A}$ (and thereby in the definition of the operator A as well) is not clear. It becomes evident if we omit this condition 2, define $\mathfrak{L}$ as consisting of all those in $\mathfrak{H}$ which satisfy condition 1, and then "define" the operator

$$Au = -u'' + \kappa\delta(x)u \quad , \quad u \text{ in } \mathfrak{L} .$$

Then $-u'' + \kappa\delta(x)u = \lambda u$, $-\infty < x < \infty$ is simply Schrödinger's equation of a particle moving along the x-axis in a field

whose potential is given by $\kappa\delta(x)$. We can now understand immediately, why a point eigenvalue exists for $\kappa < 0$ and not for $\kappa \geq 0$.

However this is not a proper definition of a linear operator on $\mathcal{L}$, because the symbolic function $\delta(x)$ is involved. We prefer to say that the form

$$A(v,u) = \int_{-\infty}^{\infty} \bar{v}'u\,dx + \kappa\,\overline{v(0)}u(0)$$

is defined for u, v in $\mathcal{L}$. Obviously $A(v,u)$ on $\mathcal{L}$ is an extension--even a trivial one--of the form $(v,Au) = -\int_{-\infty}^{\infty} \bar{v}u''dx$ on $\mathfrak{A}$.

Consider the similar example

$$Au = -(u_{xx} + u_{yy} + u_{zz}) + \kappa\,\delta(x,y,z)u = \lambda u\ , \quad -\infty < x,y,z < \infty$$

in a suitable subspace of the Hilbert space

$$\mathfrak{H}: \quad \iiint_{-\infty}^{\infty} |f(x,y,z)|^2 d\tau < \infty\ , \qquad (g,f) = \iiint_{-\infty}^{\infty} \bar{g}\,f\,d\tau\ .$$

Again we replace this improper operator by the well defined Hermitian form

$$A(v,u) = \iiint_{-\infty}^{\infty} (\bar{v}_x u_x + \bar{v}_y u_y + \bar{v}_z u_z)d\tau + \kappa\,\bar{v}(0,0,0)u(0,0,0)$$

defined for u, v in

$\mathcal{L}$: u has continuous first derivatives in $-\infty < x,y,z < \infty$,
$$\iiint_{-\infty}^{\infty} |u|^2 d\tau < \infty, \quad \iiint_{-\infty}^{\infty} |\text{grad } u|^2 d\tau < \infty .$$

This form, or any suitable Hermitian extension, is a correct substitute for the operator $-\Delta + \kappa\delta(x,y,z)$.

We call a Hermitian form $A(v,u)$ on $\mathcal{L}_A$ *bounded below if a constant a exists such that $A(u,u) \geq a(u.u)$ for all u in $\mathcal{L}_A$*.

In our example the form is bounded below if $\kappa \geq 0$, but nevertheless it is well known that the form $\iiint(\bar{v}_x u_x + \bar{v}_y u_y + \bar{v}_z u_z)d\tau + \kappa\,\bar{v}(0,0,0)u(0,0,0)$ is a thoroughly unreasonable one.

The concept of a Hermitian form $A(v,u)$ on $\mathcal{L}_A$ is too general, whereas restriction to forms which are bounded below is too special. We shall require that the form be *closeable*.

Definition. *A Hermitian form $A(v,u)$ on $\mathcal{L}$ is said to be closeable if every sequence z_n on $\mathcal{L}$ such that $\lim_{n\to\infty} \|z_n\| = 0$ and $\lim_{n,m\to\infty} A(z_n - z_m, z_n - z_m) = 0$ satisfies the relation $\lim_{n\to\infty} A(z_n, z_n) = 0$.*

Exercise. Prove that $A(v,u) = \iiint_{-\infty}^{\infty} (\bar{v}_x u_x + \bar{v}_y u_y + \bar{v}_z u_z)d\tau + \kappa\,\bar{v}(0,0,0)u(0,0,0)$ on $\mathcal{L}$, and any Hermitian extension, are not closeable.

The word "closeable" is used to indicate the intimate relationship to the concept of a closed form at least in the case of forms that are bounded below.

A Hermitian form $A(v,u)$ bounded below on $\mathfrak{A}$ is said to closed, if each element u on $\mathfrak{H}$ for which there exists a sequence u_n on $\mathfrak{A}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{n,m\to\infty} A(v_n - v_m, u_n - u_m) = 0$ belongs to $\mathfrak{A}$ and $\lim_{n\to\infty} A(u_n - u, u_n - u) = 0$. Obviously any closed form is closeable.

We shall prove that every bounded below, closeable form $A(v,u)$ on $\mathfrak{A}$ has a trivial extension which is closed, and which is called its closure. If $A(v,u)$ on $\mathfrak{A}$ is closeable and if $B(v,u)$ on $\mathfrak{B} \supseteq \mathfrak{A}$ is a trivial extension, then the closure of $A(v,u)$ on $\mathfrak{A}$ is a trivial extension of $B(v,u)$ on $\mathfrak{B}$.

In the literature a form which has a closed trivial extension is sometimes called a form which has a closure. Using this language, we have the theorem: A form bounded below is closeable if and only if it has a closure.

To prove that a closeable form $A(v,u)$ on $\mathfrak{A}$ which is bounded below has a closure we shall again introduce the norm $\|u\|_A = (A(u,u) + a(u,u))^{\frac{1}{2}}$ for all u on $\mathfrak{A}$, where a is a real constant selected so that $A(u,u) + a(u,u) \geq (u,u)$. We then have $\|u\|_A \geq \|u\|$; we can even define the inner product $(v,u)_A = A(v,u) + a(v,u)$ and then we have $\|u\|_A = (u,u)_A^{\frac{1}{2}}$. The proof will consist in showing that the subspace $\mathfrak{A}$ can be closed with respect to the norm $\|u\|_A$. But this latter fact can even be proved if the closeable form $A(v,u)$ on $\mathfrak{A}$ is not bounded below if we introduce the norm $\|u\|_A = |A(u,u)|^{\frac{1}{2}} + \|u\|$ for which again $\|u\|_A \geq \|u\|$ holds. We give a proof which covers both cases, and we shall replace our Hilbert space $\mathfrak{H}$ by a Banach space $\mathfrak{B}$ (without restrictions on dimension, e.g. separability). We prove first the general theorem:

Suppose that $\mathfrak{B}$ is a Banach space with the norm $\|u\|$. Suppose that a norm $\|u\|_A$ is defined on the subspace $\mathfrak{A} \subseteq \mathfrak{B}$, satisfying (in addition to the usual requirements) the following two conditions:

(a) $\|u\|_A \geq \|u\|$ for u on $\mathfrak{A}$.

(b) If z_n is a sequence in $\mathfrak{A}$ for which $\lim\limits_{n\to\infty} \|z_n\| = 0$ and $\lim\limits_{n,m\to\infty} \|z_n - z_m\|_A = 0$, then $\lim\limits_{n\to\infty} \|z_n\|_A = 0$.

Call $\overline{\mathfrak{A}}$ the subspace of all u on $\mathfrak{B}$ for which a sequence u_n of $\mathfrak{A}$ exists such that $\lim\limits_{n\to\infty} \|u - u_n\| = 0$ and $\lim\limits_{m,n\to\infty} \|u_n - u_m\|_A = 0$. Then the following statements hold:

1. If u is on $\overline{\mathfrak{A}}$ and u_n, v_n are two sequences of $\overline{\mathfrak{A}}$ with $\lim\limits_{n\to\infty} \|u_n - u\| = 0$, $\lim\limits_{n\to\infty} \|v_n - u\| = 0$, $\lim\limits_{m,n\to\infty} \|u_n - u_m\|_A = 0$, and $\lim\limits_{n\to\infty} \|v_n - v_m\|_A = 0$, then $\lim\limits_{n\to\infty} \|u_n\|_A = \lim\limits_{n\to\infty} \|v_n\|_A$ exist. We call the common limit $\|u\|_A$. Thus in $\overline{\mathfrak{A}}$ there is defined a norm $\|u\|_A$ which coincides with the norm $\|u\|_A$ defined in $\mathfrak{A}$.

2. This norm $\|u\|_A$ satisfies the following: (a) $\|u\|_A \geq \|u\|$ for u on $\mathfrak{A}$ and (b) if z_n is a sequence in $\overline{\mathfrak{A}}$ with $\lim\limits_{n\to\infty} \|z_n\| = 0$ and $\lim\limits_{n,m\to\infty} \|z_n - z_m\|_A = 0$, then $\lim\limits_{n\to\infty} \|z_n\|_A = 0$.

3. $\overline{\mathfrak{A}}$ is a Banach space with the norm $\|u\|_A$.

Proof. 1. The first statement follows from the fact that for $z_n = u_n - v_n$ we get $\lim\limits_{n\to\infty} \|z_n\| = 0$ and $\lim\limits_{n,m\to\infty} \|z_n - z_m\|_A = 0$, and consequently $\lim\limits_{n\to\infty} \|z_n\|_A = 0$. Now $\big|\, \|u_n\|_A - \|v_n\|_A \big| \leq \|z_n\|_A$,

and hence $\lim_{n\to\infty} \left| \|u_n\|_A - \|v_n\|_A \right| = 0$. The convergence of the sequence $\|u_n\|_A$ follows from $\left|\|u_n\|_A - \|u_m\|_A\right| \leq \|u_n - u_m\|_A \to 0$ as $n,m \to \infty$. Therefore $\lim_{n\to\infty} \|u_n\|_A = \lim_{n\to\infty} \|v_n\|_A$ exist.

2. For each u on $\bar{\mathfrak{A}}$ there exists a sequence u_n on $\mathfrak{A}$ such that $\lim_{n\to\infty} \|u - u_n\|_A = 0$.

Proof. Obviously a sequence u_n on $\mathfrak{A}$ exists with $\|u - u_n\| = 0$ and $\lim_{m,n\to\infty} \|u_n - u_m\|_A = 0$. We have $\lim_{m\to\infty} \|u_n - u_m\|_A = \|u_n - u\|_A$ according to statement 1 of our theorem. To a given $\varepsilon > 0$ choose N such that $\|u_n - u_m\|_A \leq \varepsilon$ for all $n,m \geq N$. Then $\|u_n - u\|_A \leq \varepsilon$ for $n \geq N$ which is the statement.

3. Let z_n be a sequence on $\bar{\mathfrak{A}}$ with $\lim_{n\to\infty} \|z_n\| = 0$, $\lim_{n,m\to\infty} \|z_n - z_m\|_A = 0$. Then we have $\lim_{n\to\infty} \|z_n\|_A = 0$ too.

Proof. Let $f_{n,k}$ be a sequence on $\mathfrak{A}$ with $\lim_{k\to\infty} \|f_{n,k} - z_n\|_A = 0$, $n = 1,2,\ldots$. Then, for all $k \geq k_n$ we have $\|f_{n,k} - z_n\|_A \leq \frac{1}{n}$. Call $u_n = f_{n,k_n}$. We get $\|u_n - z_n\|_A \leq \frac{1}{n}$ and $\|u_n - u_m\|_A \leq \|u_n - z_n\|_A + \|u_m - z_m\|_A + \|z_n - z_m\|_A \leq \frac{1}{n} + \frac{1}{m} + \|z_n - z_m\|_A \to 0$, $n,m \to \infty$. From $\|u_n - z_n\| \leq \|u_n - z_n\|_A \leq \frac{1}{n}$ it follows that $\|u_n\| \leq \|z_n\| + \frac{1}{n} \to 0$. Thus u_n is a sequence on $\mathfrak{A}$ with $\lim_{n\to\infty} \|u_n\| = 0$ and $\lim_{m,n\to\infty} \|u_n - u_m\|_A = 0$ from which it follows that $\lim_{n\to\infty} \|u_n\|_A = 0$. But $\|z_n\|_A \leq \|u_n - z_n\|_A + \|u_n\|_A \leq \frac{1}{n} + \|u_n\|_A \to 0$ then $\lim_{n\to\infty} \|z_n\|_A = 0$.

4. Suppose that x_n is a sequence on $\bar{\mathfrak{A}}$ such that $\lim\limits_{m,n\to\infty} \|x_n - x_m\|_A = 0$. Then there exists an element x on $\bar{\mathfrak{A}}$ such that $\lim\limits_{n\to\infty} \|x_n - x\|_A = 0$.

Proof. In any case an element x on $\mathfrak{B}$ exists with $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. We shall prove that x is in $\bar{\mathfrak{A}}$. Since x_n is on $\bar{\mathfrak{A}}$ a sequence u_n of $\mathfrak{A}$ exists with $\|u_n - x_n\|_A \le \frac{1}{n}$. This was proved in step 2. Therefore $\|u_n - x_n\| \le \frac{1}{n}$ and $\lim\limits_{n\to\infty} \|u_n - x\| = 0$. Furthermore $\|u_n - u_m\|_A \le \|u_n - x_n\|_A + \|x_m - u_m\|_A + \|x_n - x_m\|_A \to 0$, $n,m \to \infty$. Thus x belongs to $\bar{\mathfrak{A}}$. The sequence $z_n = x_n - x$ is in $\bar{\mathfrak{A}}$ with $\lim\limits_{n\to\infty} \|z_n\| = 0$, $\lim\limits_{n,m\to\infty} \|z_n - z_m\|_A = 0$. Therefore $\lim\limits_{n\to\infty} \|z_n\|_A = 0$ at $\lim\limits_{n\to\infty} \|x_n - x\|_A = 0$. This completes the proof of the theorem.

It is obvious that our condition (b) is not only sufficient but also necessary to be able to close a subspace $\mathfrak{A}$ of a Banach space $\mathfrak{B}$ in which a metric $\|u\|_A \ge \|u\|$ is given to a subspace $\bar{\mathfrak{A}}$ of $\mathfrak{B}$.

We return to our Hilbert space $\mathfrak{H}$ and prove:

Every form A(v,u) on $\mathfrak{A}$ which is closeable and bounded below has one and only one trivial extension $\bar{A}(v,u)$ on $\bar{\mathfrak{A}}$ which is closed; this is called the closure of A(v,u) on $\mathfrak{A}$.

Proof. Call $\bar{\mathfrak{A}}$ the set of all u of the Hilbert space $\mathfrak{H}$ for which there exists a sequence u_n in $\mathfrak{A}$ such that $\lim\limits_{n\to\infty} \|u_n - u\| = 0$ and $\lim\limits_{n,m\to\infty} A(u_n - u_m, u_n - u_m) = 0$. Choose a constant a such that $A(u,u) + a(u,u) \ge (u,u)$ for u in $\mathfrak{A}$, and define

$\|u\|_A = (A(u,u) + a(u,u))^{\frac{1}{2}}$. Obviously $\bar{\mathfrak{A}}$ coincides with the set of all u in $\mathfrak{H}$ for which a sequence u_n exists with $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{m,n\to\infty} \|u_n - u_m\|_A = 0$. From the above theorem we know, that $\bar{\mathfrak{A}}$ is a (complete) Banach space under the norm $\|u\|_A = \lim_{n\to\infty} \|u_n\|_A$. Thus, to a pair u, v on $\bar{\mathfrak{A}}$ there exist a pair of sequences u_n, v_n with $\lim_{n\to\infty} \|u_n - u\|_A = 0$ and $\lim_{n\to\infty} \|v_n - v\|_A = 0$. We have

$$|A(v_n,u_n) - A(v_m,u_m)| = |A(v_n - v_m,u_n) + A(v_m,u_n - u_m)|$$

$$= |(v_n - v_m,u_n)_A - a(v_n - v_m,u_n) + (v_m,u_n - u_m)_A - a(v_m,u_n - u_m)|$$

$$\leq \|v_n - v_m\|_A \|u_n\|_A + |a| \|v_n - v_m\| \|u_n\| + \|v_m\|_A \|u_n - u_m\|_A$$

$$+ |a| \|v_m\| \|u_n - u_m\| \to 0 , \quad n,m \to \infty .$$

If $\tilde{u}_n$, $\tilde{v}_n$ are any other pair of sequences such that $\lim_{n\to\infty} \|\tilde{u}_n - u\|_A = 0$ and $\lim_{n\to\infty} \|\tilde{v}_n - u\|_A = 0$ we get $\lim_{n\to\infty} A(\tilde{v}_n,\tilde{u}_n) = \lim_{n\to\infty} A(v_n,u_n)$. This makes it possible to define $\bar{A}(v,u) = \lim_{n\to\infty} A(v_n,u_n)$ for each v, u on $\bar{\mathfrak{A}}$. Obviously $\bar{A}(v,u)$ on $\mathfrak{A}$ is a trivial extension of A(v,u) on $\mathfrak{A}$.

Let A'(v,u) on $\mathfrak{A}' \supseteq \mathfrak{A}$ be another closed trivial extension of A(v,u) on $\mathfrak{A}$. To each pair v, u of $\mathfrak{A}'$ there exist sequences u_n, v_n on $\mathfrak{A}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$, $\lim_{n\to\infty} \|v_n - v\| = 0$, $\lim_{n,m\to\infty} A(u_n - u_m, u_n - u_m) = 0$,

$\lim_{n,m\to\infty} A(v_n - v_m, v_n - v_m) = 0$, and $\lim_{n\to\infty} A(v_n, u_n) = A'(v,u)$.

This shows that u, v are elements of $\bar{\mathfrak{A}}$ and that $\bar{A}(v,u) = A(v,u)$ for u, v on $\mathfrak{A}' \subseteq \bar{\mathfrak{A}}$. We need to show $\mathfrak{A}' = \bar{\mathfrak{A}}$. Take any element u on $\bar{\mathfrak{A}}$, and select a sequence u_n on $\mathfrak{A}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{n,m\to\infty} \|u_n - u_m\|_A = 0$. Such a sequence is also in $\mathfrak{A}'$ and $\lim_{n,m\to\infty} A(u_n - u_m, u_n - u_m)$ $= \lim_{n,m\to\infty} A'(u_n - u_m, u_n - u_m) = 0$. Since $A'(v,u)$ on $\mathfrak{A}'$ is closed u is in $\mathfrak{A}'$, hence $\mathfrak{A}' \supseteq \bar{\mathfrak{A}}$. This completes the proof.

It is a well-known fact that every operator A on $\mathfrak{A}$ which is bounded below defines an associated form (v,Au) on $\mathfrak{A}$ which is bounded below and has a closure. Therefore we expect that (v,Au) on $\mathfrak{A}$ is closeable. But the latter is even true if the Hermitian operator A on $\mathfrak{A}$ is not bounded below.
H. C. Kranzer proved:

If A on a subspace $\mathfrak{A}$ of $\mathfrak{H}$ is a Hermitian operator, then the form (v,Au) on $\mathfrak{A}$ is closeable.

What we must prove is this: If z_n is a subsequence of $\mathfrak{A}$ with $\lim_{n\to\infty} \|z_n\| = 0$, $\lim_{n,m\to\infty} (z_n - z_m, A(z_n - z_m)) = 0$ then $\lim_{n\to\infty} (z_n, Az_n) = 0$. Let $u_k = z_{n_k}$, $k = 1,2,\ldots$, $\lim_{k\to\infty} u_k = 0$ be a subsequence for which $(u_k, Au_k) \geq 0$ for all $k = 1,2,\ldots$ or $(u_k, Au_k) \geq 0$ for all $k = 1,2,\ldots$ holds. If we have proved $\lim_{k\to\infty} (u_k, Au_k) = 0$ for each such subsequence then $\lim_{n\to\infty} (z_n, Az_n) = 0$ is proved too. To a given $\varepsilon > 0$ choose a such that $|(u_k - u_j, A(u_k - u_j))| < \varepsilon$ if $k, j > K$. Then $|(u_k, Au_k) + (u_j, Au_j)| < \varepsilon + |(u_j, Au_k)| + |(Au_k, u_j)|$. To a given

$k > K$ choose $j = j(\varepsilon,k)$ such that $|(u_j,Au_k)| + |(Au_k,u_j)| < \varepsilon$. This is possible since $\lim_{j\to\infty} \|u_j\| = 0$. Thus $|(u_k,Au_k) + (u_j,Au_j)| < 2\varepsilon$ for every $k > K$ if $j = j(\varepsilon,k)$. But $|(u_k,Au_k) + (u_j,Au_j)| \geq |(u_k,Au_k)|$ and therefore $|(u_k,Au_k)| < 2\varepsilon$ for every $k > K$. This means $\lim_{k\to\infty} (u_k,Au_k) = 0$ and the assertion is established.

It can be proved:[*] Corresponding to any bounded below, closed form $A(v,u)$ on $\mathfrak{L}$ there is one and only one hypermaximal operator $\bar{A}$ on $\bar{\mathfrak{A}} \subseteq \mathfrak{L}$ such that (1) $(v,\bar{A}u) = A(v,u)$ for v on $\mathfrak{L}$, u on $\bar{\mathfrak{A}}$ and (2) the form $A(v,u)$ on $\mathfrak{L}$ is a trivial extension of (v,Au) on $\bar{\mathfrak{A}}$.

If we begin with a Hermitian operator A on $\mathfrak{A}$, which is bounded below, we can define the form (v,Au) on $\mathfrak{A}$. It is closeable and bounded below. Therefore a trivial extension exists which is closed; called $A(v,u)$ on $\mathfrak{G}$. Because of the theorem cited above there exists a unique hypermaximal $\bar{A}$ on $\bar{\mathfrak{A}} \subseteq \mathfrak{G}$ such that (1) $(v,\bar{A}u) = A(v,u)$ for v on $\mathfrak{L}$, u on $\bar{\mathfrak{A}}$ and (2) $A(v,u)$ on $\mathfrak{G}$ is a trivial extension of (v,Au) on $\mathfrak{A}$. This hypermaximal extension $\bar{A}$ on $\bar{\mathfrak{A}}$ of the Hermitian (bounded below) operator A on $\mathfrak{A}$ is called the Friedrichs-extension of A on $\mathfrak{A}$.

Since we prefer to deal with essential self-adjoint operators (rather than hypermaximal operators), we shall give a slightly more general definition.

An operator B on $\mathfrak{B}$ is called a Friedrichs extension of a Hermitian operator A on $\mathfrak{A}$ which is bounded below if it is an essentially self-adjoint extension of A on $\mathfrak{A}$ such that the form (v,Bu) on $\mathfrak{B}$ is a trivial extension of the form (v,Au) on $\mathfrak{A}$.

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[*] K. O. Friedrichs, Spektraltheorie halbbeschränkter Operatoren und Anwendungen auf die Spektralzerlegung von Differentialoperatoren, Math. Ann., vol. 109 (1934), pp. 465-487, especially Satz 7 and Satz 9 (pp. 478-480).

89. A criterion for regularity involving forms instead of operators.

Consider an open region D with boundary Γ in the x,y,z-space which contains the origin $x = y = z = 0$; and consider the Hilbert space

$$\mathfrak{H}: \quad \int_D |f(x,y,z)|^2 d\tau < \infty \quad , \quad (g,f) = \int_D \bar{g} f \, d\tau$$

and the subspace

$\dot{\mathfrak{D}}$: $u(x,y,z)$ has continuous fourth derivatives in D and vanishes identically in neighborhoods of the origin of every boundary point of D. Then the operator $A_o u = \Delta \Delta u$ is a Hermitian operator on $\dot{\mathfrak{D}}$ and, since $(u, A_o u) = \int_D |\Delta u|^2 d\tau$, A_o on $\dot{\mathfrak{D}}$ is bounded below. This operator is not essentially self-adjoint, but it has an essentially self-adjoint Friedrichs extension whose spectrum is discrete, $\lambda_1, \lambda_2, \ldots$ with $\lim_{n \to \infty} \lambda_n = \infty$. It is characterized by certain boundary conditions at Γ and at the origin.

We consider on $\dot{\mathfrak{D}}$ the perturbed operator $A(\varepsilon) = A_o + \varepsilon A_1$, with

$$A_1 u = \frac{s(x,y,z)}{r^4} u$$

where $s(x,y,z)$ is a continuous bounded function in D, and

$$|s(x,y,z) < s_o \quad \text{in D} \quad .$$

Can we expect the eigenvalue equation

$$\Delta \Delta u + \varepsilon \frac{s(x,y,z)}{r^4} u = \lambda u$$

to have point eigenvalues which are power series in ε? First we observe that $A(\varepsilon) = A_0 + \varepsilon A_1$ in $\dot{\mathcal{D}}$ is not essentially self-adjoint. Does it have a Friedrichs extension? This will be the case if $A(\varepsilon)$ in $\dot{\mathcal{D}}$ is bounded below. For u in $\dot{\mathcal{D}}$ we have

$$|(u,A_1u)| \leq \int_D \frac{s_0}{r^4}|u|^2 d\tau \leq \frac{16s_0}{9} \int_D |\triangle u|^2 d\tau = \frac{16s_0}{9}(u,A_0u) .$$

Hence

$$(u,\ A(\varepsilon)u) \geq (u,A_0u) - \frac{16s_0|\varepsilon|}{9}(u,A_0u)$$

and consequently $A(\varepsilon)$ on $\dot{\mathcal{D}}$ is bounded below for $|\varepsilon| \leq \frac{9}{16s_0}$ and so has a Friedrichs extension. Now our question makes sense: Does this extension have point eigenvalues which are power series in ε?

The answer is yes, because this essentially self-adjoint extension of $A(\varepsilon)$ in $\dot{\mathcal{D}}$ is *regular*. This follows immediately from the following

Criterion 4. *Suppose that $A_0, A_1, A_2, \ldots$ are Hermitian operators on a subspace $\dot{\mathcal{D}}$. Suppose that $A_0u + \varepsilon A_1u + \ldots = A(\varepsilon)u$ converges for each u in $\dot{\mathcal{D}}$. Suppose that A_0 on $\dot{\mathcal{D}}$ is bounded below, and that there exist three non-negative constants p, a, b such that*

$$|(u,A_\nu u)| \leq p^{\nu-1}(a(u,u) + b(u,A_0u)) \qquad \nu = 1,2,\ldots$$

holds for all u in $\dot{\mathcal{D}}$. Then for real ε near $\varepsilon = 0$ the operator $A(\varepsilon)$ in $\dot{\mathcal{D}}$ is bounded below and its Friedrichs extension is regular.

In our example we have $A_2 = 0$, $A_3 = 0, \ldots,$ and we can choose $p = 0$, $a = 0$, $b = \frac{16s_0}{9}$.

We indicate the proof of this criterion. The bounded below form $(v, A_0 u)$ on $\dot{\mathfrak{D}}$ has a closure $A_0(v,u)$ on $\mathfrak{F} \subseteq \mathfrak{H}$. From the inequalities it follows that trivial extensions $A_\nu(v,u)$ on $\mathfrak{F}$ of the form $(v, A_\nu u)$ on $\dot{\mathfrak{D}}$ exist. For small $|\varepsilon|$ we have $A_\varepsilon(v,u) = \sum_{\nu=0}^{\infty} \varepsilon^\nu A_\nu(v,u)$ for u, v in $\mathfrak{F}$. Now the the statements of Criterion 4 are contained in Rellich [4] Satz 3.

If one compares Criterion 4 with Criterion 3 (page 105), one might be led to ask the following. Suppose that A on $\mathfrak{A}$ and S on $\mathfrak{A}$ are positive, Hermitian operators, i.e. $(u, Au) \geq 0$, $(u, Su) \geq 0$ for u in $\mathfrak{A}$. We then ask: Does the inequality $(u, Su) \leq (u, Au)$ in $\mathfrak{A}$ imply the inequality $(Su, Su) \leq (Au, Au)$? The answer is no, even in the case of a finite dimensional Hilbert space. It is well known that from $S \leq A$ for positive-definite Hermitian matrices one cannot conclude $S^2 \leq A^2$.

We ask further: Does $(Su, Su) \leq (Au, Au)$ in $\mathfrak{A}$ imply $(u, Su) \leq (u, Au)$ in $\mathfrak{A}$? The answer is yes provided we make the additional assumption that A on $\mathfrak{A}$ is essentially self-adjoint. This was proved by E. Heinz . If we do not make this additional assumption, the answer is no. This can be seen from the example $Au = -u''$ on $\mathfrak{A}$, where $\mathfrak{A}$ is the set of functions u with u, u', u'' continuous in $0 \leq x < \infty$, $u \equiv 0$ near $x = 0$ and $x = \infty$, and $Su = \frac{3}{4} x^{-2} u$ in $\mathfrak{A}$. As we have shown in §7, we have $(Su, Su) \leq (Au, Au)$. But $(u, Su) \leq (u, Au)$ for u in $\mathfrak{A}$ would mean

$$\frac{3}{4} \int_0^\infty x^{-2} |u|^2 dx \leq \int_0^\infty |u'|^2 dx$$

whereas

$$\inf_{\substack{u \not\equiv 0 \\ u \text{ in } \mathfrak{a}}} \frac{\int_0^\infty |u'|^2 dx}{\int_0^\infty x^{-2}|u|^2\, dx} = \frac{1}{4} \quad .$$

§10. Are the perturbed eigenfunctions complete?

Suppose that $A(\varepsilon)$ in $\mathfrak{A}(\varepsilon)$ is a regular operator, which for $\varepsilon = 0$ has a discrete spectrum $\lambda_1, \lambda_2, \ldots$ with orthonormal and complete eigenelements $\phi_1, \phi_2, \ldots$. We know that to each λ_n, ϕ_n there exist power series

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon \lambda_n^{(1)} + \ldots$$

$$\phi_n(\varepsilon) = \phi_n + \varepsilon \phi_n^{(1)} + \ldots$$

which converge for $|\varepsilon|$ small enough. We ask: Are the eigenelements $\phi_n(\varepsilon)$ complete for small $|\varepsilon| \neq 0$? This question is not well posed. It might happen that the convergence radius of $\phi_n(\varepsilon)$ tends to zero as $n \longrightarrow \infty$. In that case it would not make sense to talk of the whole set $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ for a fixed $\varepsilon \neq 0$, even if $|\varepsilon|$ is small. Therefore we shall assume that the $\phi_n(\varepsilon)$ as well as the $\lambda_n(\varepsilon)$ can all be defined by analytic continuation in a fixed interval $-\rho < \varepsilon < \rho$. Are then the elements $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$, all defined in $-\rho < \varepsilon < \rho$, a complete set of eigenelements?

In spite of the fact that $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ is supposed to be regular, the answer is no!

Consider the Hilbert space

$$\mathfrak{H}: \qquad \int_0^1 |f(x)|^2 dx < \infty \quad , \qquad (g,f) = \int_0^1 \bar{g}\, f\, dx$$

and the subspace

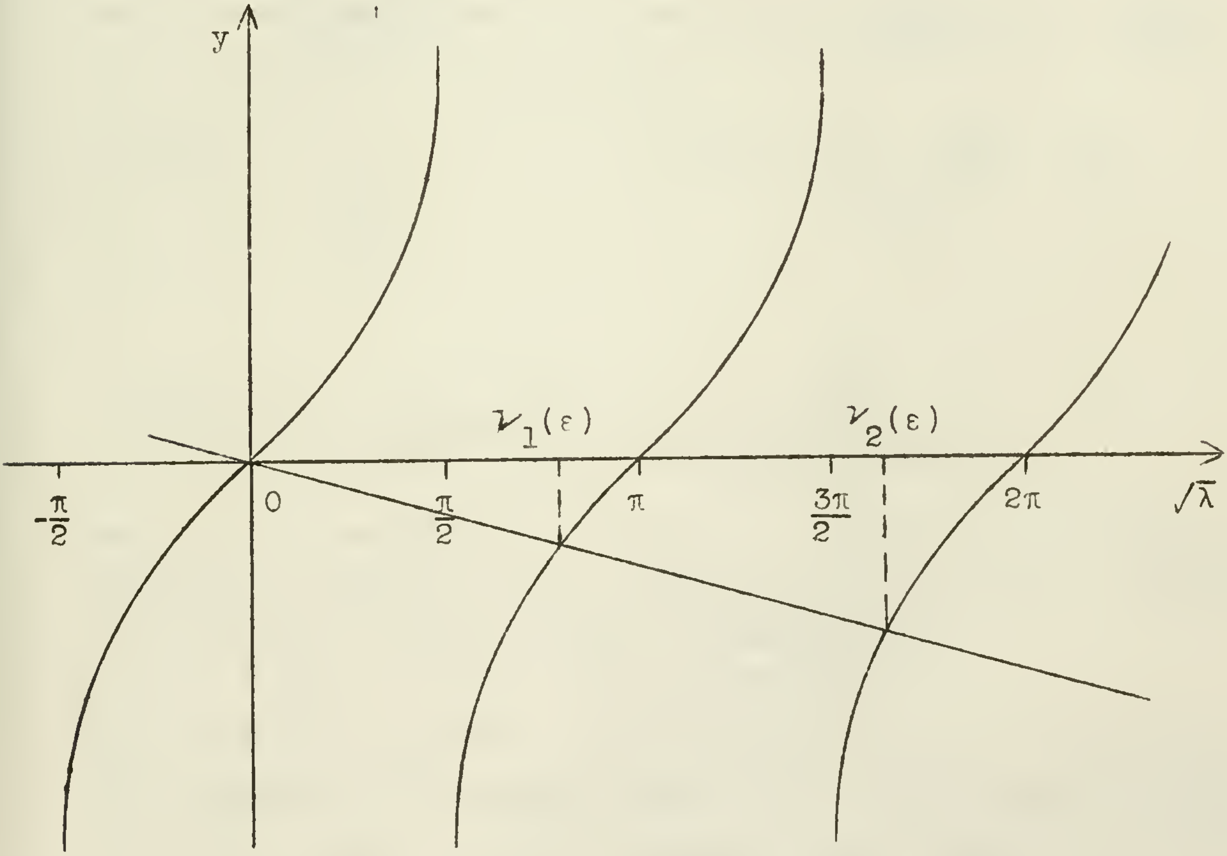
$\mathfrak{A}(\varepsilon)$: 1. u, u', u'' continuous in $0 \leq x \leq 1$

2. $u(0) = 0, \quad \varepsilon u'(1) + u(1) = 0$.

As we know (page 103) we have for $A(\varepsilon)u = -u''$ on $\mathfrak{A}(\varepsilon)$

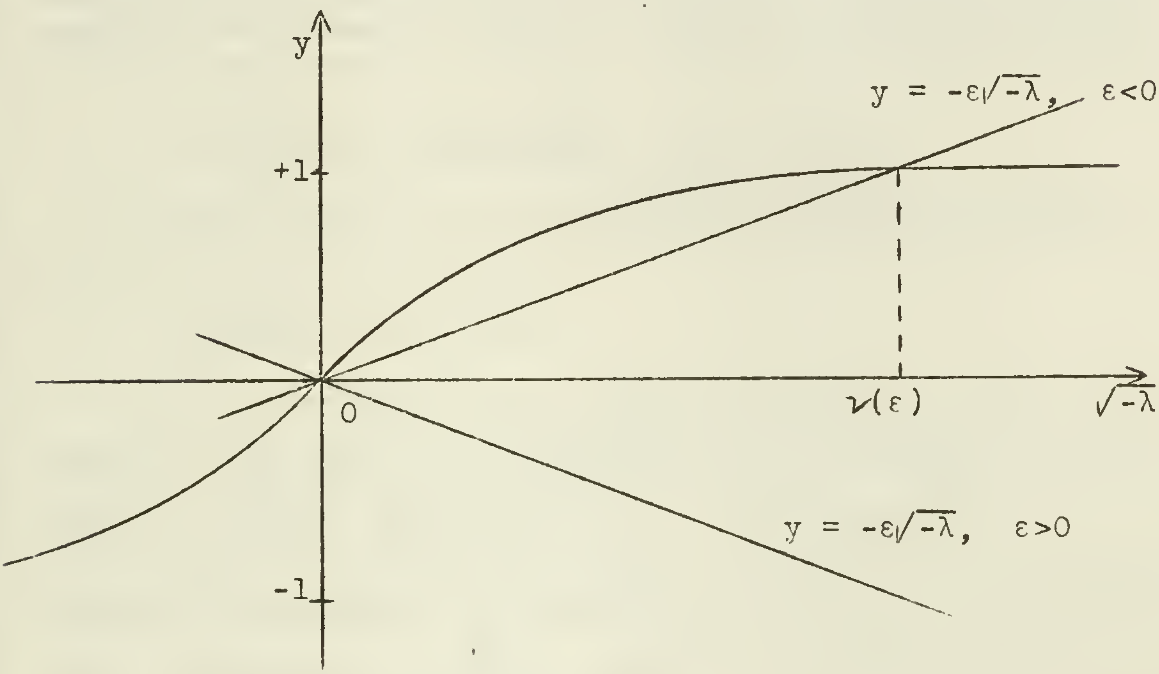
$$A^{-1}(\varepsilon)f = \int_0^1 G(x,y;\varepsilon)f(y)dy \; ; \quad G(x,y;\varepsilon) = \tfrac{1}{2}(x+y) - \tfrac{1}{2}|x-y| - \frac{xy}{1+\varepsilon} ,$$

for all continuous f; hence $A(\varepsilon)$ in $\mathfrak{A}(\varepsilon)$ is regular. We construct the spectrum of $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ which consists of point eigenvalues only. We restrict ourselves to $-1 < \varepsilon < 1$; then $\lambda = 0$ is not an eigenvalue. If λ is a positive eigenvalue there exists a function $u(x) \not\equiv 0$ in $\mathfrak{A}(\varepsilon)$ such that $-u'' = \lambda u$, $u(x) = a \sin(\sqrt{\lambda}\, x)$, $\tan\sqrt{\lambda} = -\varepsilon\sqrt{\lambda}$



Graphs of $y = \tan\sqrt{\lambda}$ and $y = -\varepsilon\sqrt{\lambda}$

If $\varepsilon = 0$, we have the unperturbed eigenvalues $\lambda_n = n^2\pi^2$, $n = 1,2,\ldots$, and the normalized eigenfunctions $\phi_n = \sqrt{2}\sin n\pi x$. If $\varepsilon \neq 0$ we get $\lambda_n = \nu_n^2(\varepsilon)$ with the (non-normalized) eigenfunctions $\phi_n(\varepsilon) = \sqrt{2}\sin \nu_n(\varepsilon)x$. If $\varepsilon \geq 0$, then the eigenfunctions are complete because in this case there is no negative eigenvalue λ for such an eigenvalue λ should be a solution of the equation $\tanh\sqrt{-\lambda} = -\varepsilon\sqrt{-\lambda}$



Graphs of $y = \tanh\sqrt{-\lambda}$ and $y = -\varepsilon\sqrt{-\lambda}$

which has no solution $\lambda \neq 0$ if $\varepsilon \geq 0$. But if $\varepsilon < 0$ we have exactly one solution $\lambda^*(\varepsilon) = -\nu^2(\varepsilon) \neq 0$ if $0 < \varepsilon < 1$. To this λ^* belongs the (non-normalized) eigenelement $\phi_*(x) = \sinh\ (\nu(\varepsilon)x)$. The eigenfunctions $\phi_n(\varepsilon) = \sqrt{2}\sin\ (\nu_n(\varepsilon)x)$ are not complete for $0 < \varepsilon < 1$, and the eigenvalue $\lambda^*(\varepsilon)$ which happens to be the lowest eigenvalue of $A(\varepsilon)$ in $\mathfrak{A}(\varepsilon)$ is not a convergent power series near $\varepsilon = 0$.

We are interested in conditions which guarantee the completeness of the perturbed eigenelements. We first present:

Definition of a regular discrete spectrum. *Let $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ be a Hermitian operator for $\rho_1 < \varepsilon < \rho_2$. We say it has a regular discrete spectrum in this interval if functions $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$, and elements $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ of $\mathfrak{A}(\varepsilon)$ exist, all regular in a neighborhood of every ε of this interval, such that for every ε of $\rho_1 < \varepsilon < \rho_2$ the following holds:*

(1) $A(\varepsilon)\phi_n(\varepsilon) = \lambda_n(\varepsilon)\phi_n(\varepsilon)$, $n = 1,2,\ldots$

(2) $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ *is a complete orthonormal system*

(3) $\lim\limits_{n \to \infty} |\lambda_n(\varepsilon)| = \infty$.

If $A(\varepsilon)$ on $\mathfrak{A}(\varepsilon)$ has a regular discrete spectrum, this does not mean that all eigenvalues of the operator must be regular analytic functions of ε. Of course, one can arrange the spectrum in such a way that one obtains non-regular eigenvalues. If the $\lambda_n(\varepsilon)$, $n = 1,2,\ldots$ are bounded below, then $\mu(\varepsilon) = \underset{n=1,2,3,\ldots}{\text{Min}} \lambda_n(\varepsilon)$ is in general not a regular function of ε near $\varepsilon = 0$. But $\mu(\varepsilon)$ is an eigenvalue, i.e. an element $\psi(\varepsilon) \neq 0$ on $\mathfrak{A}(\varepsilon)$ exists with $A(\varepsilon)\psi(\varepsilon) = \mu(\varepsilon)\psi(\varepsilon)$. Thus $\mu(\varepsilon)$ need not be one of the $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$ in $\rho_1 < \varepsilon < \rho_2$. But if we make the additional assumption, that $\mu(\varepsilon)$ is regular in a neighborhood of every ε_0 of the closed interval $\rho_0 - \delta \leq \varepsilon \leq \rho_0 + \delta$ where $\rho_1 < \rho_0 - \delta < \rho_0 + \delta < \rho_2$, then one can prove that $\mu(\varepsilon)$ must coincide with one of the functions $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$ in the whole interval $|\varepsilon - \rho_0| \leq \delta$. This follows from the fact that the points of this interval are not denumerable. Indeed, for each ε' of the interval, $\mu(\varepsilon')$ must be one of the numbers

$\lambda_1(\varepsilon'), \lambda_2(\varepsilon'), \ldots$. If $\mu(\varepsilon)$ would coincide in $|\varepsilon - \rho_0| \leq \delta$ with each of the functions $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$ only for a finite set of points ε', then the interval $|\varepsilon - \rho_0| \leq \delta$ would consist only of a denumerable set of points, which is not the case. Thus $\mu(\varepsilon) = \lambda_j(\varepsilon)$ for infinitely many points ε of the interval $|\varepsilon - \rho_0| \leq \delta$. Since $\mu(\varepsilon)$ and $\lambda_j(\varepsilon)$ are analytic functions, we have $\mu(\varepsilon) \equiv \lambda_j(\varepsilon)$.

One can prove

Theorem 1. *For real ε near $\varepsilon = 0$ let $A(\varepsilon)$ be a Hermitian operator on a subspace $\mathfrak{A}$ which is independent of ε. Suppose that $A(\varepsilon)u$ is a regular element for every u of $\mathfrak{A}$ and small $|\varepsilon|$. Suppose that $A(0)$ on $\mathfrak{A}$ is hypermaximal and has a discrete spectrum. Then $A(\varepsilon)$ has a regular discrete spectrum near $\varepsilon = 0$.*

The proof of this theorem can be derived from the following theorem about completely continuous operators:

Theorem 2. *Let $R_0, R_1, \ldots$ be bounded Hermitian operators in a (infinite dimensional) Hilbert space $\mathfrak{H}$. Suppose that R_0 is completely continuous and that there exists a constant $k \geq 0$ such that $\|R_n u\| \leq k^n \|R_0 u\|$, $n = 1,2,\ldots$. Then the following statements hold: $R(\varepsilon) = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots$ is a Hermitian, bounded operator in $\mathfrak{H}$, regular in $-\frac{1}{k} < \varepsilon < \frac{1}{k}$. In $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$, the functions $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$ and the elements $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ are all convergent power series in neighborhoods of each ε of this interval such that for $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$ the following is true:*

(1) *$R(\varepsilon)\phi_n(\varepsilon) = \lambda_n(\varepsilon)\phi_n(\varepsilon)$, $n = 1,2,\ldots$*

(2) *$\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ is an orthonormal complete system*

(3) $\lim\limits_{n \to \infty} |\lambda_n(\varepsilon)| = 0$.

Every $\lambda_n(\varepsilon)$ is either different from zero or identically zero, and those $\phi_n(\varepsilon)$ which belong to the identically-zero eigenvalues are independent of ε.

In the case that one knows $(u, R_o u) \geq 0$ for all u in $\mathfrak{H}$ one can prove

Theorem 3. *Let $R_o, R_1, \ldots$ be bounded Hermitian operators in an infinite dimensional Hilbert space $\mathfrak{H}$. Suppose that R_o is completely continuous and that there exists a constant $k \geq 0$ such that $|(u, R_n u)| \leq k^n (u, R_o u)$. Then for $R(\varepsilon) = R_o + \varepsilon R_1 + \ldots$ the conclusions of Theorem 2 hold.*

Remarks. Obviously the infinitely many conditions $\|R_n u\| \leq k^n \|R_o u\|$, $n = 1, 2, \ldots$, of Theorem 1 or $|(u, R_n u)| \leq k^n (u, R_o u)$ of Theorem 2 can be replaced by the conditions:

(1) $R(\varepsilon) = R_o + \varepsilon R_1 + \ldots$ is a bounded regular operator for complex ε with $|\varepsilon| < \rho$, Hermitian for real ε.

(2) For all complex ε with $|\varepsilon| < \rho$ there exists a constant M such that $\|R(\varepsilon)u\| \leq M \|R_o x\|$ or $|(u, R(\varepsilon)u)| \leq M(u, R_o u)$.

If $R(\varepsilon)$ is an integral operator, $R(\varepsilon)u = \int_0^1 K(x,y;\varepsilon)u(y)dy$ and $K(x,y;\varepsilon)$ is a power series in ε converging uniformly for all x,y of the interval $0 \leq x,y \leq 1$ and $K(x,y;\varepsilon) = \overline{K(y,x;\varepsilon)}$ for real ε then our two conditions become

$$\int_0^1 \left| \int_0^1 K(x,y;\varepsilon)u(y)dy \right|^2 dx \leq M^2 \int_0^1 \left| \int_0^1 K(x,y;0)u(y)dy \right|^2$$

and

$$\left|\int_0^1 \int_0^1 K(x,y;\varepsilon)u(x)\overline{u(y)}dxdy\right| \leq M \int_0^1 \int_0^1 K(x,y;0)u(x)\overline{u(y)}dxdy$$

respectively.

Take as an example the kernel $K(x,y;\varepsilon) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| - \frac{\varepsilon}{1+\varepsilon}\, xy$, $0 \leq x,y \leq 1$ which was Green's function of the problem $-u'' = \lambda u$, $u(0) = 0$, $u'(1) + \varepsilon u(1) = 0$.

We obtain

$$I = \int_0^1 \int_0^1 K(x,y;0)u(x)\overline{u(y)}dy = 2 \operatorname{Re} \int_0^1 dx\, u(x) \int_x^1 x\, \bar{u}(y)dy \quad ,$$

and introducing $\int_x^1 u(y)dy = v(x)$, we find

$$I = 2 \operatorname{Re} \int_0^1 dx(-v'(x)) \int_x^1 x\, \bar{u}(y)dy = 2 \operatorname{Re} \int_0^1 v(x)\, x\, \bar{u}(x)dx$$

$$= - \int_0^1 x(v(x)\bar{v}'(x) + \bar{v}(x)v'(x))dx = \int_0^1 |v(x)|^2 dx \quad .$$

On the other hand we have

$$\left|\int_0^1 \int_0^1 K(x,y;\varepsilon)u(x)\overline{u(y)}\, dxdy\right| \leq \left|\int_0^1 \int_0^1 \frac{-\varepsilon}{1+\varepsilon}\, xy\; u(x)\overline{u(y)}dxdy\right| + I$$

$$\leq \frac{1}{1-q} \left|\int_0^1 x\, u(x)dx\right|^2 dx + I$$

if ε is a complex number and $|\varepsilon| \leq q < 1$. From

$$\left|\int_0^1 x\,u(x)dx\right|^2 = \left|\int_0^1 x\,v'(x)dx\right|^2 = \left|\int_0^1 v(x)dx\right|^2 \leq \int_0^1 |v(x)|^2 dx = I,$$

we obtain

$$\left|\int_0^1 \int_0^1 K(x,y;\varepsilon)u(x)\overline{u(y)}dxdy\right|$$

$$\leq \left(\frac{1}{1-q}+1\right) \int_0^1 \int_0^1 K(x,y;0)u(x)\overline{u(y)}dxdy,$$

which is exactly the inequality $|(u,R(\varepsilon)u)| \leq M(u,R(0)u)$ with $M = \frac{1}{1-q}+1$ valid for all complex ε with $|\varepsilon| \leq q$.

Proof of Theorem 2. The operator $R(\varepsilon) = R_o + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots$ can be defined for complex ε with $|\varepsilon| < \frac{1}{k}$ and $\|R(\varepsilon)u\| \leq \|R_o u\| \frac{1}{1-|\varepsilon|k}$, thus

$$\|R(\varepsilon)u\| \leq 2\|R_o u\| \quad \text{if} \quad |\varepsilon| \leq \frac{1}{2k} \quad .$$

In the following we denote by q_1, q two fixed numbers with $0 < q_1 < q < \frac{1}{2k}$. From

$$R(\varepsilon) = \frac{1}{2\pi i} \int_{|t|=\frac{1}{2k}} \frac{R(t)}{t-\varepsilon} dt ,$$

it follows that

$$R'(\varepsilon) = \frac{1}{2\pi i} \int_{|t|=\frac{1}{2k}} \frac{R(t)}{(t-\varepsilon)^2} dt$$

if $|\varepsilon| \leq q_1$. Hence $\|R'(\varepsilon)u\| \leq C\|R_o u\|$ if $|\varepsilon| < q_1$. From

$$\|R(\varepsilon)u\| \geq \|R_o u\| - \|R_o u\| \, (|\varepsilon|k + |\varepsilon|^2 k^2 + \ldots)$$

$$\geq \left(1 - \frac{|\varepsilon|k}{1 - |\varepsilon|k}\right)\|R_o u\| = \frac{1 - 2|\varepsilon|k}{1 - |\varepsilon|k}\|R_o u\| \geq p\|R_o u\| \quad \text{for } |\varepsilon| \leq q_1 ,$$

where $p > 0$ is a constant. Thus we have

$$\|R'(\varepsilon)u\| \leq C'\|R(\varepsilon)u\| \quad \text{for } |\varepsilon| \leq q_1 \quad .$$

From $\|R(\varepsilon)u\| \leq 2\|R_o u\|$ in $|\varepsilon| \leq q_1$ follows that $R(\varepsilon)$ is completely continuous for $|\varepsilon| \leq q_1$. Thus for ε of the interval $-q_1 \leq \varepsilon \leq q_1$ the operator has a pure point spectrum and each eigenelement of $R(\varepsilon)$ belonging to the eigenvalue $\lambda = 0$ is an eigenvalue of $R(\varepsilon)$ with the eigenvalue $\lambda = 0$ and vice versa.

Let $\lambda(\varepsilon)$ be an eigenvalue, regular analytic in the neighborhood of every ε_o of the interval $\rho_1 < \varepsilon < \rho_2$, where $-q_1 < \rho_1 < \rho_2 < q_1$. Let $\varepsilon = \eta$ be a point of the interval $\rho_1 < \varepsilon < \rho_2$ with $\lambda(\eta) \neq 0$. Let $\phi(\varepsilon)$ be a normalized eigenelement belonging to $\lambda(\varepsilon)$, regular in a neighborhood of $\varepsilon = \eta$. From $R(\eta)\phi(\eta) = \lambda(\eta)\phi(\eta)$ we have $|\lambda(\eta)| \leq 2C$ if $\|R_o u\| \leq C\|u\|$. From $\lambda'(\eta) = (\phi(\eta), R'(\eta)\phi(\eta))$ we get $|\lambda'(\eta)| \leq C'\|R(\eta)\phi(\eta)\|$ thus $|\lambda'(\eta)| \leq C'|\lambda(\eta)|$.

Now let us start with an eigenelement $\lambda \neq 0$ of $R(\varepsilon_o)$ $-q_1 < \varepsilon_o < q_1$. We know that in a neighborhood of $\varepsilon = \varepsilon_o$ a regular eigen-value $\lambda(\varepsilon)$ of $R(\varepsilon)$ exists. We want to show that $\lambda(\varepsilon)$ has an analytic continuation over $-q_1 \leq \varepsilon \leq q_1$ and that $\lambda(\varepsilon) \neq 0$ in $-q_1 \leq \varepsilon \leq q_1$. Otherwise a θ exists (we may assume $\theta > \varepsilon_o$) such that $\lambda(\varepsilon)$ is regular and $\neq 0$ in $\varepsilon_o \leq \varepsilon < \theta$ whereas $\lambda(\varepsilon)$ either is not regular at θ or vanishes for $\theta = 0$. Since $|\lambda(\varepsilon)| \leq 2C$, we have two cases: 1) a sequence ε_n exists, $\varepsilon_o \leq \varepsilon_n < \theta$, $\lim\limits_{n \to \infty} \varepsilon_n = \theta$, and $\lim\limits_{n \to \infty} \lambda(\varepsilon_n) = \mu \neq 0$, or 2) $\lambda(\varepsilon) \to 0$ for $\varepsilon \to \theta$, $\varepsilon_o \leq \varepsilon < \theta$. In the first case $\lambda(\varepsilon)$ can be continued analytically over $\varepsilon = \theta$

because of our general theorem about eigenvalues of finite multiplicity. The second case is impossible. From $|\lambda'(\varepsilon)| \leq C'|\lambda(\eta)|$ and $\lim\limits_{\varepsilon \to \theta} \lambda(\varepsilon) = 0$ it follows that $\lambda(\varepsilon) = 0$ for $\varepsilon_o \leq \varepsilon \leq \theta$ in a well-known way.

Thus all the eigenvalues $\lambda(\varepsilon) \neq 0$ for $\varepsilon = 0$ can be continued over the whole interval $q_1 \leq \varepsilon \leq q_2$ without vanishing. One can construct eigenelements $\phi(\varepsilon)$ belonging to each of these $\lambda(\varepsilon)$. Add to these $\phi(\varepsilon)$ a complete set of eigenelements of the eigenvalue $\lambda = 0$ of R_o, which does not depend on ε. Call $\phi_n(\varepsilon)$, $n = 1,2,\ldots$ all together. Obviously $\lambda_n(\varepsilon)$ and $\phi_n(\varepsilon)$, $n = 1,2,\ldots$ are regular in $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$. We have to show the completeness of the $\phi_n(\varepsilon)$ for an arbitrary point ε_o with $-\frac{1}{2k} < \varepsilon_o < \frac{1}{2k}$. If they were not complete a $\psi \neq 0$ would exist with $R(\varepsilon_o)\psi = \mu\psi$ and ψ orthogonal to all $\phi_n(\varepsilon)$, $n = 1,2,\ldots$. Obviously $\mu \neq 0$. Construct an eigenelement regular analytic together with its eigenvalue, which becomes μ for $\varepsilon = \varepsilon_o$. This can be continued over $-\frac{1}{2k} < \varepsilon_o < \frac{1}{2k}$, call it $\mu(\varepsilon)$. To this $\mu(\varepsilon)$ we construct an eigenelement $\psi(\varepsilon)$, $\|\psi(\varepsilon)\| = 1$, regular analytic at each point of the interval $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$. (To see that this is possible, see Rellich [5], page 467.) Obviously $\mu(\varepsilon_o)$ must coincide with one of the numbers $\lambda_n(\varepsilon_o)$, $n = 1,2,\ldots$. Let us assume

$\mu(\varepsilon) = \lambda_1(\varepsilon) \equiv \lambda_2(\varepsilon) \equiv \ldots \equiv \lambda_s(\varepsilon)$, but $\lambda_r(\varepsilon) \not\equiv \lambda_1(\varepsilon)$ if $r > s$.

Then $\psi(\varepsilon) = \sum\limits_{i=1}^{s} (\phi_s(\varepsilon),\psi(\varepsilon))\phi_s(\varepsilon)$ first in a neighborhood of $\varepsilon = 0$, then in $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$ and therefore

$\psi(\varepsilon_o) = \sum\limits_{i=1}^{s} (\phi_s(\varepsilon_o),\psi(\varepsilon_o))\phi_i(\varepsilon_o)$. But $\psi(\varepsilon_o)$ is orthogonal to all $\phi_n(\varepsilon_o)$ thus we have the contradiction $\psi(\varepsilon_o) = 0$ and $\|\psi(\varepsilon_o)\| = 1$. This proves the theorem.

Theorem 3 can be proved similarly, and Theorem 1 is a consequence of Theorem 2 (see Rellich [5]).

Appendix

(Proof of Lemma on page 50)

Lemma: Suppose that $F(x,y)$ is a convergent power series with real coefficients in some neighborhood of $x = y = 0$, and suppose that $F(x,y) \geq 0$ for real x,y in that neighborhood. Then there exist two convergent power series $G(x,y)$ and $H(x,y)$ with real coefficients such that in some neighborhood of $x = y = 0$,

$$F(x,y) = [G(x,y)]^2 + [H(x,y)]^2 \quad .$$

Proof. Weierstrass' preparation theorem asserts that, except for the trivial case $F \equiv 0$, after a real linear homogeneous transformation, the function F can be written in the form EF_1, where the function E is regular near the origin and $E(0,0) \neq 0$ and F_1 is a polynomial of the form

$$(1) \qquad y^n + \sum_{\nu=1}^{n} f_\nu(x) y^{n-\nu}$$

with coefficients $f_\nu(x)$ which are regular functions near $x = 0$ and $f_\nu(0) = 0$. Any proof of the preparation theorem demonstrates at the same time that E and f_ν are real in the real case. If we put $x = 0$ and consider small positive values of y, then we observe that necessarily $E(0,0)$ is positive; and if we consider small negative values of y, then we can conclude that n is an even integer, say $n = 2m$. We claim that F is the product of two conjugate factors $K = G + iH$ and $\overline{K} = G - iH$. ($\overline{K}$ denotes the power series whose coefficients are the complex conjugates of those of K; thus e.g. $\overline{K}(\overline{x},\overline{y}) = \overline{K(x,y)}$.) To prove this assertion it is enough to prove that F_1 can be written as the product of two conjugate factors, because if $F_1 = K\overline{K}$ then $F = K\sqrt{E} \cdot \overline{K}\sqrt{E}$, and $\sqrt{E}$ is regular and real near the origin (for E is regular and positive for real values of x and y near the origin).

Now a polynomial of the form (1) can be factored into prime factors of the same form in a unique way. If two of these prime factors are conjugates--they may even coincide in which case F_1 would have a real double prime factor--then $F_1 = P\overline{P}F_2$; and if $F_2 = Q\overline{Q}$ is the product of two conjugate factors, then so is $F_1 = PQ \cdot \overline{PQ}$. Now factorization into a pair of conjugate prime factors preserves the property of F_1; namely, each factor is real and non-negative for real values of the variables. Therefore to prove our claim it suffices to prove it in the case that F_1 has no pair of conjugate prime factors. Since $F_1 = \overline{F}_1$, if P is a prime factor of F_1 then so is $\overline{P}$; and consequently $P = \overline{P}$ is real. Thus we need only consider the case where the prime factors of F_1 are all real and distinct. But as we shall immediately establish, this case can occur, under the assumptions of the problem, only in a trivial manner, i.e. where $F_1 \equiv 1$.

We shall show that a product of distinct real prime factors changes sign in every real neighborhood of the origin. In a deleted neighborhood of the origin, the zeros of any prime factor are simple and different from those of any other prime factor. Hence if we find a real zero of a prime factor arbitrarily near the origin, then at that point the product must change sign. The zeros of a prime factor P(x,y) of the form (1) can be represented as follows:

$$P(\xi^n, y) = \prod_{\nu=0}^{n-1} [(y - \phi(e^{\frac{2\pi i}{n}\nu} \xi)]$$

where $\phi(\xi)$ is regular near $\xi = 0$ and $\phi(0) = 0$. Since P is real, for ξ real we have

$$\prod_{\nu=0}^{n-1} [y - \phi(e^{\frac{2\pi i}{n}\nu} \xi)] = \prod_{\nu=0}^{n-1} [y - \overline{\phi}(e^{-\frac{2\pi i}{n}\nu} \xi)] \quad ;$$

and consequently

$$\phi(\xi) = \bar{\phi}(e^{-\frac{2\pi i}{n}k}\xi)$$

for some k with $0 \leq k \leq n-1$, and hence

$$\phi(e^{\frac{\pi i}{n}k}\xi) = \bar{\phi}(e^{-\frac{\pi i}{n}k}\xi) \quad .$$

If we put

$$\psi(\xi) = \phi(e^{\frac{\pi i}{n}k}\xi) \quad ,$$

then we have

$$\psi(\xi) = \bar{\psi}(\xi)$$

for ξ real. Now

$$\prod_{\nu=0}^{n-1}[y - \psi(e^{-\frac{2\pi i}{n}\nu}\xi)] = \prod_{\nu=0}^{n-1}[y - \phi^{\frac{\pi i}{n}(2\nu+k)}\xi)]$$

$$= P((e^{\frac{\pi i}{n}k}\xi)^n, y)$$

$$= P((-1)^k\xi^n, y) \quad .$$

Therefore for sufficiently small real ξ the pair of values $((-1)^k\xi^n, \psi(\xi))$ is a real zero lying arbitrarily near the origin.

This completes the proof.

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